JOINT PROGRAM EXAM
REAL ANALYSIS
FALL 1997

Instructions: Instructions: You may take up to $3\frac{1}{2}$ hours to complete the exam. Completeness in your answers is very important. Justify your steps by referring to theorems by name, when appropriate, or by providing a brief theorem statement. An essentially complete and correct solution to one problem will gain more credit, than solutions to two problems each of which is "half correct".

Notation: Throughout the exam, $\mathbb{R}$ stands for the set of real numbers.

1. For a measurable subset E of $\mathbb{R}^n$, prove or disprove:
   (a) If E has Lebesgue measure 0, then its closure has Lebesgue measure 0.
   (b) If the closure of E has Lebesgue measure 0, then E has Lebesgue measure 0.
2. (i) Prove that convergence in L^1 implies convergence in measure. (Give a precise statement, then prove it.)
   (ii) Is the converse true?
3. Denote the ternary Cantor set in $[0,1]$ by C. Let $f:[0,1]\to\mathbb{R}$ be defined by
$$f(x)=\begin{cases}0, & x\in C\\ x^2, & x\notin C.\end{cases}$$
   Prove that f is a Lebesgue measurable function.
4. Show that the hyperelliptic integral
$$\int_2^\infty \frac{x dx}{\sqrt{(x^2-\varepsilon^2)(x^2-1)(x-2)}}$$
   converges to the elliptic integral
$$\int_2^\infty \frac{dx}{\sqrt{(x^2-1)(x-2)}}$$
   when ε tends to zero.

5. Show that
$$\left(\int_0^1 \frac{x^{1/2}dx}{(1-x)^{1/3}}\right)^3 \leq \frac{8}{5}.$$
6. Evaluate
$$\sum_{n=0}^{\infty} \int_{1/2}^{\infty} (1-\mathrm{e}^{-t})^n \mathrm{e}^{-t^2} dt.$$
7. Either give an example of the specified mathematical object or quote a theorem that proves that no such object can exist.
(i) A sequence f_n of functions converging to 0 uniformly on $[0,1]$, but such that $\int_0^1 f_n dm \geq 1$ for all n (here m denotes the Lebesgue measure).
(ii) A sequence f_n of functions converging to 0 uniformly on $\mathbb{R}$, but such that $\int_{\mathbb{R}} f_n dm \geq 1$ for all n (here m denotes the Lebesgue measure).
(iii) A sequence f_n of positive functions converging pointwise to a function f such that $\liminf \int_0^1 f_n < \int_0^1 f$.
8. Let Σ denote the σ-algebra of subsets of $\mathbb{R}$ generated by the set $\{0\}$ (i.e. Σ is the smallest σ-algebra in $2^{\mathbb{R}}$, which contains $\{0\}$).
(i) Prove that every $E \in \Sigma$ is Lebesgue measurable.
Let μ denote the restriction of the Lebesgue measure m (on $\mathbb{R}$) to Σ [i.e. $\forall E \in \Sigma : \mu(E) = m(E)$]. It is easy to show (you do not have to!) that μ is a measure on Σ.
(ii) Determine the dimension of $L^2(\mathbb{R}, \mu)$.
Help: Σ is so 'small' that one can list its elements.