UNIVERSITY OF ALABAMA SYSTEM
JOINT DOCTORAL PROGRAM IN APPLIED
MATHEMATICS
JOINT PROGRAM EXAMINATION
Linear Algebra and Numerical Linear Algebra

TIME: THREE AND ONE HALF HOURS

May, 2001

Instructions: Do 7 of the 8 problems for full credit. Be sure to indicate which 7 are to be graded. Include all work. Write your student ID number on every page of your exam.
1. Let $V$ be the vector space of polynomials of degree at most 2 with complex coefficients and consider the linear transformation $D : V \rightarrow V$, $y \mapsto y'$. Find the eigenvalues of $D$, their geometric and algebraic multiplicities, and the minimal and characteristic polynomials of $D$. Determine a basis of $V$ such that the matrix of $D$ with respect to this basis is in Jordan canonical form.

2. Let $A \in \mathbb{R}^{n \times n}$ be given, singular. Use the Schur Theorem to show that, for any $\epsilon > 0$, there is a non-singular matrix $A_\epsilon$ such that $\|A - A_\epsilon\|_2 \leq \epsilon$. Can a similar statement be proved for an arbitrary defective matrix $A$ and a non-defective matrix $A_\epsilon$?

3. Suppose $A \in \mathbb{C}^{m \times n}$ has rank $n$ and $b \in \mathbb{C}^n$. Prove that the block linear system

$$
\begin{bmatrix}
I_{m \times m} & A \\
A^* & 0_{n \times n}
\end{bmatrix}
\begin{bmatrix}
r \\
x
\end{bmatrix}
= 
\begin{bmatrix}
b \\
0_{n \times n}
\end{bmatrix}
$$

has a unique solution $(r, x)^T$ where $r \in \mathbb{C}^m$ and $x \in \mathbb{C}^n$. Show that $r$ and $x$ must be the residual and solution of the least squares problem for minimizing $\|b - Ax\|_2$.

4. Let $A$ and $B$ be two linear transformations such that $AB - BA = I$, the identity. Show that $A^kB - BA^k = kA^{k-1}$, for all integers $k > 1$.

5. Given a non-singular $A \in \mathbb{R}^{n \times n}$, show that

(a) $AA^T$ and $A^TA$ have the same eigenvalues, all positive, but (generally) different eigenvectors,

(b) if these eigenvalues are arranged in descending order of magnitude, the condition number $\kappa_2(A) = \sqrt{\lambda_1/\lambda_n}$,

(c) the condition

$$
\frac{\|\delta A\|}{\|A\|} < \frac{1}{\kappa(A)},
$$

for any norm, guarantees that the perturbed matrix $(A + \delta A)$ is non-singular.

6. (a) Let $V$ be a finite-dimensional subspace of $\mathbb{C}^n$. Prove that for any $x \in \mathbb{C}^n$, there exists $p \in V$ and $q \in \mathbb{C}^n$ such that $x = p + q$ and $(y, q) = 0$ for all $y \in V$. (b) Let $\mathcal{V}$ be an inner product space and $\mathcal{W}$ a finite dimensional subspace of $\mathcal{V}$. For $x \in \mathcal{V}$, show that the orthogonal projection of $x$ onto $\mathcal{W}$ is the unique vector in $\mathcal{W}$ closest to $x$. 
7. Let $A \in \mathbb{R}^{n \times n}$ be a symmetric matrix with eigenvalues such that $|\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq \ldots \geq |\lambda_n| > 0$. Suppose $z \in \mathbb{R}^n$ with $z^T x_1 \neq 0$, where $Ax_1 = \lambda_1 x_1$. Prove that, for some constant $C$,

$$\lim_{k \to \infty} \frac{A^k z}{\lambda_1^k} = C x_1$$

and describe a reliable algorithm, based on this result, for computing $\lambda_1$ and $x_1$. Explain how the calculation should be modified to obtain (a) $\lambda_n$ and (b) the eigenvalue closest to 2.

8. Let $T : V \to W$, $U : W \to V$ be linear transformations such that $(UT)(x) = x, \forall x \in V$ where $\dim V = \dim W < \infty$. Without assuming invertibility, establish the following:

(a) $T$ is $1 - 1$;
(b) $T$ is onto;
(c) $T^{-1}$ exists and $T^{-1} = U$;
(d) If $A$ and $B$ are square matrices with $AB = I$, then both $A$ and $B$ are invertible and $A^{-1} = B, B^{-1} = A$. 