Instructions: Completeness in answers is very important. Justify your steps by referring to theorems by name where appropriate. Include all work. Full credit will accrue from answering 5 of the 7 problems given. Indicate which solutions you want to be graded if you work on more than 5 problems.
1. Let $P_2(\mathbb{R})$ be the set of polynomials of degree less than or equal to 2, defined over the real line, and define
\[ T : P_2(\mathbb{R}) \rightarrow P_2(\mathbb{R}) \]
according to
\[ T(p) = q \]
where
\[ q(x) = (1 - x)p'(x) \]
(a) Find a basis for the range and the null space of this transformation. Is it invertible?
(b) Find the characteristic polynomial, minimal polynomial and Jordan canonical form of $T$.

2. Let $V$ be the vector space of all complex valued polynomials defined over the half line $[0, \infty)$.
(a) Show that
\[ \langle f, g \rangle := \int_{0}^{\infty} f(x)\overline{g(x)}e^{-x} \, dx \]
is a complex inner product on $V$.
(b) Find an orthonormal set $\{f_0, f_1\}$ in $V$ such that $\text{span}\{e_0, e_1\} = \text{span}\{f_0, f_1\}$, where $e_0(x) = 1$ and $e_1(x) = x$.

3. Suppose that $A$ is a complex normal matrix. Prove that
(a) $A$ and $A^*$ have the same eigenvectors;
(b) if $x$ and $y$ are two eigenvectors of $A$ corresponding to distinct eigenvalues, then $x$ and $y$ are orthogonal;
(c) if $A$ is also upper-triangular, then $A$ must be diagonal.

4. (a) Give a definition of the condition number, $K(A)$, of a matrix $A$ with respect to the infinity norm.
(b) Compute the condition number of
\[ A = \begin{pmatrix} 1 & 2 \\ 1.01 & 2 \end{pmatrix} \]
(c) Show that if \( B \) is singular, then
\[
\frac{1}{K(A)} \leq \frac{\|A - B\|}{\|A\|}
\]

(d) Use (c) to estimate \( K(A) \), where \( A \) is the matrix given in (b), and compare to the solution obtained in (b).

5. (a) Give an explanation of what is meant by the least squares solution of \( Ax = b \), where \( A \in \mathbb{R}^{m \times n} \) and \( b \in \mathbb{R}^m \).

(b) Find the least squares solution of the system
\[
\begin{pmatrix}
-1 & 1 \\
1 & -1 \\
1 & 1 \\
-1 & -1
\end{pmatrix}
\begin{pmatrix}
x \\
y
\end{pmatrix}
= \begin{pmatrix}
2 \\
0 \\
2 \\
0
\end{pmatrix}.
\]

(c) Also compute the norm of the minimal residual vector.

6. Let \( A \in \mathbb{R}^{n \times n} \) be given, and assume that all leading principal submatrices of \( A \) are nonsingular. Show that there exists a unique upper triangular matrix \( U \) and a unique unit lower triangular matrix \( L \), such that \( A = LU \). If \( A \) is nonsingular, but not all the leading principal submatrices are nonsingular, what is the result now? (You don’t have to prove this one, just explain it.)

7. Let \( A \in \mathbb{R}^{n \times n} \) be given, symmetric, and assume that the eigenvalues of \( A \) satisfy
\[
|\lambda_1| > |\lambda_2| \geq \ldots \geq |\lambda_{n-1}| \geq |\lambda_n|.
\]
Let \( z \in \mathbb{R}^n \) be given. Under what conditions on \( z \) does the following hold, theoretically? (Be sure to actually show that it holds!)
\[
\lim_{k \to \infty} \frac{z^T A^{k+1} z}{z^T A^k z} = \lambda_i
\]
Under what conditions on \( z \) does this hold, as a practical matter? Explain fully for full credit.