UNIVERSITY OF ALABAMA SYSTEM

Joint Doctoral Program in Applied Mathematics
Joint Program Exam in Linear Algebra and Numerical Linear Algebra

May 2014

Exam Rules:

- This is a closed book examination. Once the exam begins, you have three and one half hours to do your best. **You are required to do seven of the eight problems for full credit.**
- Each problem is worth 10 points; parts of problems have equal value unless otherwise specified.
- Justify your solutions: cite theorems that you use, provide counter examples for disproof, give explanations, and show calculations for numerical problems.
- Begin each solution on a new page and write the last four digits of your university student ID number, and problem number, on every page. Please write only on one side of each sheet of paper.
- If you are asked to prove a theorem, do not merely quote that theorem as your proof; instead, produce an independent proof.
- The use of calculators or other electronic gadgets is not permitted during the exam.
- Write legibly using dark pencil or pen

1. Let $A \in \mathbb{C}^{n\times n}$, and let

$$\det(\lambda I_n - A) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0$$

denote the characteristic polynomial of A (here I_n is the $n \times n$ identity matrix).

   (a) Prove that if A is invertible, then

$$A^{-1} = -\frac{1}{a_0}\left[A^{n-1} + a_{n-1}A^{n-2} + \cdots + a_2 A + a_1 I_n\right].$$

   (b) Use this formula to compute A^{-1} for

$$A = \begin{bmatrix} -2 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

2. Find a reduced SVD for the matrix $A = \begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix}$. Use it to solve the linear least squares (LS) problem

$$\min_x \|b - Ax\|_2, \qquad A = \begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix}, \quad b = \begin{bmatrix} 10 \\ 5 \\ 5 \end{bmatrix}$$

3. Let $A \in \mathbb{C}^{n\times n}$ be a nonsingular upper Hessenberg matrix. Prove that if $A = LU$ is an LU decomposition of A, then $L = (l_{ij})$ has zeros below its first subdiagonal, i.e., $l_{ij} = 0$ for all $i \geq j + 2$.

4. Let $A \in \mathbb{C}^{n\times n}$ be a normal matrix, i.e., $AA^* = A^*A$. Use Schur decomposition to prove that there exists a unitary matrix Q and a diagonal matrix D such that $A = QDQ^*$.

5. Let $V \subset \mathbb{C}^n$ be a k-dimensional vector subspace, $k < n$. Let $\{q_1, \ldots, q_k\}$ be an orthonormal basis in V. Let $Q \in \mathbb{C}^{n\times k}$ denote the matrix whose columns are $q_1, \ldots, q_k$. Prove that

$$P = QQ^*$$

is the orthogonal projection of $\mathbb{C}^n$ onto V, i.e., the projection onto V along $V^{\perp}$.

6. (a) Let $A \in \mathbb{C}^{m\times n}$. Suppose $B = QA$, where $Q \in \mathbb{C}^{m\times m}$ is a unitary matrix. Show that A and B have the same singular values.

   (b) Let $B \in \mathbb{C}^{m\times n}$ be obtained by interchanging two rows of a matrix $A \in \mathbb{C}^{m\times n}$. Show that A and B have the same singular values.

   (c) Let $B \in \mathbb{C}^{m\times n}$ be obtained by interchanging two columns of a matrix $A \in \mathbb{C}^{m\times n}$. Show that A and B have the same singular values.

7. Suppose a matrix $A \in \mathbb{C}^{m\times n}$ with $m \geq n$ has the form $A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$, where A_2 is a nonsingular matrix of dimension $n \times n$ and A_1 is an arbitrary matrix of dimension $(m-n) \times n$. Let A^+ be the pseudoinverse of A defined as $A^+ = (A^*A)^{-1}A^*$. Prove that $\|A^+\|_2 \leq \|A_2^{-1}\|_2$.

8. Let $A \in \mathbb{C}^{m\times n}$ with $m \geq n$, and let $A = \hat{Q}\hat{R}$ be a reduced QR decomposition of A. Suppose $\hat{R}$ has exactly k nonzero diagonal entries ($k < n$). What does this imply about the rank of A? You should choose one of the following answers and prove that your answer is correct:

   (a) $\operatorname{rank} A = k$;

   (b) $\operatorname{rank} A \geq k$;

   (c) $\operatorname{rank} A \leq k$.

   (No need to show that the other two answers are wrong.)