University of Alabama System
Joint Ph.D Program in Applied Mathematics

Joint Program Exam: Linear Algebra and Numerical Linear Algebra

September 2012

Instructions:

• This is a closed book examination. Once the exam begins, you have three and one half hours to do your best. You are required to do seven of the eight problems for full credit.

• Each problem is worth 10 points; parts of problems have equal value unless otherwise specified.

• Justify your solutions: cite theorems that you use, provide counter examples for disproof, give explanations, and show calculations for numerical problems.

• Begin each solution on a new page and write the last four digits of your university student ID number, and problem number, on every page. Please write only on one side of each sheet of paper.

• The use of calculators or other electronic gadgets is not permitted during the exam.

• Write legibly using dark pencil or pen.
1. Let

\[ A = \begin{bmatrix}
1 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0
\end{bmatrix}, \]

where \( A \) is to be considered as a matrix over \( \mathbb{C} \).

(a) Determine the minimal and characteristic polynomials of \( A \) and the Jordan form of \( A \).

(b) Determine all generalized eigenvectors of \( A \) and a basis \( B \) of \( \mathbb{C}^4 \) with respect to which the operator \( T_A : x \rightarrow Ax \) has Jordan form.

2. Let \( P_n \) be the vector space of all polynomials of degree at most \( n \) over \( \mathbb{R} \). Define \( T : P_n \rightarrow P_n \) by \( T(p(x)) = xp'(x) - p(x) \).

(a) (3 pts) Show that \( T \) is a linear transformation on \( P_n \).

(b) (7 pts) Find the Null(T) and Range(T).

3. (a) Let \( A \in \mathbb{R}^{n \times n} \) be a symmetric \( n \times n \) matrix. Prove that \( A \) is positive definite, i.e., \( x^T Ax > 0 \) for all \( x \in \mathbb{R}^n \setminus \{0\} \), if and only if all the eigenvalues of \( A \) are positive.

(b) \[ A = \begin{bmatrix}
2 & 0 & 0 \\
0 & 3 & -1 \\
0 & -1 & 3
\end{bmatrix}. \]

Put \( V = \mathbb{R}^3 \). Define the map \( * : V \times V \rightarrow \mathbb{R} \) by \( u * v = u^T Av \) for all \( u, v \in V \). Prove that \( * \) is an inner product on \( V \).

(c) Use the inner product from above and the Gram-Schmidt orthogonalization process to find an orthonormal basis for \( V \).

4. Let \( A \in \mathbb{R}^{n \times n} \) be a symmetric positive definite matrix. Show that its 2-norm condition number can be computed as follows:

\[ \text{cond}_2(A) = \frac{\max_{\|x\|_2=1} \langle Ax, x \rangle}{\min_{\|x\|_2=1} \langle Ax, x \rangle} \]
5. Let $U \in \mathbb{R}^{n \times n}$ be an upper triangular matrix with non-zero diagonal entries $u_{ii} \neq 0$. Show that its 2-norm condition number satisfies
\[
\text{cond}_2(U) \geq \frac{\max_{1 \leq i \leq n} |u_{ii}|}{\min_{1 \leq i \leq n} |u_{ii}|}
\]

6. Let $A, B \in \mathbb{C}^{n \times n}$ be two Hermitian matrices, and assume that $B$ is positive definite. A number $\lambda \in \mathbb{C}$ is called a generalized eigenvalue for the pair $(A, B)$ if $Ax = \lambda Bx$ for some non-zero vector $x \in \mathbb{C}^n$, in which case $x$ is called generalized eigenvector. Prove that all the generalized eigenvalues are real and there exists a basis of generalized eigenvectors.

7. The matrix $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ represents clockwise rotation by 90 degrees. Show that $A$ can be factored into the product of two Householder reflectors, i.e. $A = Q_1Q_2$, where $Q_i = I - 2v_iv_i^*$ and $||v_i||_2 = 1$ for $i = 1, 2$ by determining the vectors $v_1, v_2$.

8. Let $A \in \mathbb{C}^{n \times n}$. The power method to find the dominant eigenvalue and the corresponding eigenvector of the matrix $A$ is as follows:

For $k = 1, 2, \ldots$, do

\begin{align*}
\text{set } w^{(k)} &= Av^{(k-1)} \\
\text{find the smallest integer } p \text{ with } 1 \leq p \leq n \text{ and } ||w^{(k)}||_\infty = |w_p^{(k)}| \\
\text{set } \mu_k &= w_p^{(k)} \\
\text{set } v^{(k)} &= w^{(k)}/\mu_k.
\end{align*}

Assume the following three conditions: (i) $A$ has $n$ linearly independent eigenvectors, $x_k$, $1 \leq k \leq n$, (ii) The eigenvalues $\lambda_k$ satisfy
\[
|\lambda_1| > |\lambda_2| \geq \cdots \geq |\lambda_n|
\]
and (iii) The vector $v^{(0)} \in \mathbb{C}^n$ is such that $v^{(0)} = \sum_{k=1}^n \xi_k x_k$ and $\xi_1 \neq 0$.

Prove that
\[
\lim_{k \to \infty} \mu_k = \lambda_1, \quad |\lambda_1 - \mu_k| = O\left(\frac{|\lambda_2|}{|\lambda_1|} \right).
\]