Linear Algebra and Numerical Linear Algebra

Time: Three and One Half Hours

September 12, 2002

Instructions: Do 7 of the 8 problems for full credit. Be sure to indicate which 7 are to be graded. Include all work for full credit. Write your social security number on each of your answer sheet.
A matrix $A \in \mathbb{C}^{n \times n}$ is said to be skew Hermitian if $A^* = -A$.

(a) Prove that if $A$ is skew Hermitian and $B$ is unitary equivalent to $A$, then $B$ is also skew Hermitian.

(b) What special form does the Shur decomposition theorem take for a skew Hermitian matrix $A$?

(c) Prove that the eigenvalues of a skew Hermitian matrix are purely imaginary, i.e. they satisfy $\bar{\lambda} = -\lambda$.

Let $A$ be a 13 $\times$ 13 complex matrix with characteristic polynomial $C_A(x) = x^7(x - i)^6$, minimal polynomial $M_A(x) = x^4(x - i)^3$, and $\dim E_0 = 3$, $\dim E_i = 2$, where $E_\lambda$ is the eigenspace corresponding to an eigenvalue $\lambda$ of $A$. Find a Jordan canonical form of $A$.

(b) Let $A$ be a 6 $\times$ 6 complex matrix with $C_A(x) = (x^2 + 1)^3$, $\dim E_i = 2$ and $\dim E_{-i} = 1$. Find the minimal polynomial of $A$.

(a) Define the condition number, $\kappa(A)$, for a nonsingular matrix $A \in \mathbb{R}^{n \times n}$, show that $\kappa(A) \geq 1$ and that $\kappa(AB) \leq \kappa(A)\kappa(B)$.

(b) Consider the linear system $Ax = b$. Let $x^*$ be the exact solution, and let $x_c$ be some computed approximate solution. Let $e = x^* - x_c$ be the error and $r = b - Ax_c$ be the residual for $x_c$. Show that

$$\left(\frac{1}{\kappa(A)}\right) \frac{\|r\|}{\|b\|} \leq \frac{\|e\|}{\|x^*\|} \leq \kappa(A) \frac{\|r\|}{\|b\|}.$$

(c) Interpret the above inequality for $\kappa(A)$ close to 1 and for $\kappa(A)$ large.

Let $A \in \mathbb{C}^{n \times n}$ have two distinct eigenvalues $\lambda_1$ and $\lambda_2$. Prove that the following three statements are equivalent:

(a) $A$ is diagonalizable,

(b) each column vector of $A - \lambda_2 I$ is in the eigenspace $E_{\lambda_1}$,

(c) each column vector of $A - \lambda_2 I$ is in the eigenspace $E_{\lambda_1}$ and each column vector of $A - \lambda_1 I$ is in the eigenspace $E_{\lambda_2}$. 

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Let $A$ be a given $n \times n$ nonsingular matrix, and assume a splitting of the form $A = M - N$, where $M$ is nonsingular. Let $x$ be the solution of the problem $Ax = b$. Consider the iteration

$$Mx^{(k+1)} = b + Nx^{(k)}.$$ 

Show that the errors $e^{(k)} = x - x^{(k)}$ satisfy a relation of the form:

$$e^{(k+1)} = Ge^{(k)}$$

and that the residuals $r^{(k)} = b - Ax^{(k)}$ satisfy a relation of the form

$$r^{(k+1)} = Hr^{(k)}$$

for appropriate matrices $G$ and $H$. How are $G$ and $H$ related? Prove that $\rho(H) < 1$ if and only if $\rho(G) < 1$, where $\rho(A)$ is the spectral radius of the matrix $A$.

Let $u \in \mathbb{R}^n$ be a given vector and

$$P = I - \frac{2}{uu^T}uu^T$$

be a Householder reflector matrix.

(a) Prove that $P$ is orthogonal.

(b) Let $x$ be given and let $x = v + w$ where $v$ lies along the vector $u$ and $w$ is orthogonal to $u$. Show that $Px = -v + w$, and explain why $P$ is called a ”reflector” matrix.

(c) For a given matrix $A$, explain briefly how to use Householder matrices to compute the decomposition $A = QR$ where $Q$ is orthogonal and $R$ is upper triangular.

Consider the 3 vectors

$$v_1 = \begin{pmatrix} 1 \\ \epsilon \\ 0 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ 0 \\ \epsilon \\ 0 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \epsilon \end{pmatrix},$$

where $\epsilon \ll 1$.

(a) Use the Classical Gram-Schmidt method to compute 3 orthonormal vectors $q_1$, $q_2$ and $q_3$, making the approximation that $1 + \epsilon^2 \approx 1$ (that is replace any term containing $\epsilon^2$ or smaller with zero, but retain terms containing $\epsilon$). Are all the $q_i$ ($i = 1, 2, 3$) pairwise orthogonal? If not, why not?

(b) Repeat (a) using the modified Gram-Schmidt orthogonalization process. Are the $q_i$ ($i = 1, 2, 3$) pairwise orthogonal? If not, why not?
Consider the matrix
\[ A = \begin{pmatrix} -2 & 11 \\ -10 & 5 \end{pmatrix}. \]

(a) Determine, a real SVD of \( A \) in the form \( A = U \Sigma V^T \).

(b) What are the 1-, 2-, \( \infty \)-, and Frobenius norms of \( A \)?

(c) Find \( A^{-1} \) not directly, but via the SVD.

(d) Find the eigenvalues \( \lambda_1, \lambda_2 \) of \( A \).

(e) Verify that \( \det A = \lambda_1 \lambda_2 \) and \( |\det A| = \sigma_1 \sigma_2 \), where \( \sigma_1 \) and \( \sigma_2 \) are singular values of \( A \).