Instructions: Completeness in answers is very important. Justify your steps by referring to theorems by name where appropriate. Include all work. Full credit will be given for correctly answering 6 of the 7 problems given. Indicate which solutions you want to be graded if you work on more than 6 problems.
1. Let $V = \mathbb{R}^{n \times n}$ be the vector space of all real $n \times n$ matrices and $T : V \to V$ be the transformation defined by $T(A) = \frac{1}{2}(A + A^T)$.

(a) Prove that $T$ is linear.
(b) Find a basis of the null space of $T$ and determine its dimension.
(c) Find a basis of the range of $T$ and determine its dimension.

2. Let $\mathcal{P}_3(\mathbb{R})$ denote the space of all polynomials of degree $\leq 3$ with real coefficients. Find $p \in \mathcal{P}_3(\mathbb{R})$ such that $p(0) = 0$ and

$$\int_{-1}^{1} (2 + 3t - p(t))^2 \, dt$$

is as small as possible.

3. (a) Find a Jordan form $J$ for $A = \begin{pmatrix} i & 1 \\ 1 & -i \end{pmatrix}$ and a nonsingular $P$ such that $P^{-1}AP = J$

(b) Prove that every complex $2 \times 2$ matrix is similar to a symmetric matrix.

4. Let $V$ be a finite dimensional inner product space with an inner product $\langle \cdot, \cdot \rangle$ and a norm $\| \cdot \| = \sqrt{\langle \cdot, \cdot \rangle}$. Let a linear operator $T : V \to V$ be self–adjoint. The Rayleigh quotient for $x \neq 0$ is defined as

$$R(x) = \frac{\langle T(x), x \rangle}{\|x\|^2}.$$

Prove that $\max_{x \neq 0} R(x)$ is the largest eigenvalue of $T$ and $\min_{x \neq 0} R(x)$ is the smallest eigenvalue of $T$. 
5. Consider the iteration: \( Q_{k+1} R_{k+1} = A Q_k \), where \( A \in \mathbb{C}^{n \times n} \) is nonsingular, \( Q_0 = I \), \( Q_k \in \mathbb{C}^{n \times n} \) is unitary, and \( R_k \in \mathbb{C}^{n \times n} \) is upper triangular. Prove that there exists an upper triangular matrix \( U_k \) such that \( Q_k = A^k U_k \) and a lower triangular matrix \( L_k \) such that \( Q_k = (A^H)^{-k} L_k \), where \( A^H \) is the conjugate transpose of \( A \).

6. Let

\[
A = \begin{pmatrix} 3 & 3 \\ 0 & 4 \\ 4 & -1 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}.
\]

(a) Use the Gram–Schmidt process to find an orthonormal basis for the column space of \( A \).

(b) Factor \( A \) into a product \( QR \), where \( Q \in \mathbb{R}^{3 \times 2} \) has an orthonormal set of column vectors and \( R \in \mathbb{R}^{2 \times 2} \) is upper triangular.

(c) Solve the least squares problem \( Ax = b \).

7. (a) Show that given an invertible matrix \( A \in \mathbb{R}^{n \times n} \), one can choose vectors \( b \in \mathbb{R}^n \) and \( \Delta b \in \mathbb{R}^n \) so that if

\[
Ax = b, \\
A(x + \Delta x) = b + \Delta b,
\]

then

\[
\frac{\|\Delta x\|_2}{\|x\|_2} = \kappa_2(A) \frac{\|\Delta b\|_2}{\|b\|_2},
\]

where \( \kappa_2(A) = \|A\|_2 \|A^{-1}\|_2 \) is the 2-condition number.

(b) Explain the significance of part (a) for the ‘optimal’ role of condition numbers in the sensitivity analysis of linear systems.