General treatment of the thermal noises in optical fibers

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A general framework based on the fluctuation-dissipation theorem has been outlined for the study of the spontaneous thermal fluctuations in optical fibers. The goal is to seek a unified scheme to analyze the two types of intrinsic noises found in fiber lasers and interferometric fiber-optic sensors, namely, the thermoconductive noise and the thermomechanical noise. Some outstanding questions in the current theories are addressed. These include: (a) the underlying relation between the thermoconductive and the thermomechanical noises, (b) the lack of a fully disclosed theory for the thermoconductive noise in passive fibers, and (c) the low-frequency restriction in the current theory of the thermomechanical noise. Specific analyses based on the proposed approach find excellent agreement with existing theories.

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I. INTRODUCTION

Fiber-optic lasers and interferometric sensors have become pervasively needed for high-sensitivity optical sensing. An important question concerning their operation is: What sets the fundamental limit of phase (or frequency) stability in these instruments? Over the last two decades, overwhelming experimental evidence has pointed to the spontaneous thermal fluctuations of optical fibers as the primary source of intrinsic phase noise [1–7]. Meanwhile, several theoretical models have been proposed in an effort to understand the underlying physics [8–15]. Most notable among them are the theories by Wanser [9], Foster *et al.* [13], and Duan [14]. Recently, Bartolo *et al.* have reported a thorough experimental study on fiber thermal noises with careful comparisons with these three theories [7]. It shows that the overall noise spectrum results from the combination of different types of spontaneous fluctuations. This finding motivates the present paper in which a more unified theoretical approach is sought to describe the different thermal noise mechanisms in optical fibers. It may prove invaluable in the discussion of intrinsic noise sources of fiber lasers and precision fiber-optic instruments.

Historically, thermal noise has been studied in both passive fibers and fiber lasers. For passive fibers, early theoretical work has focused on the thermal noise due to spontaneous local temperature fluctuations. Such noise, as will be shown later, is closely related to energy dissipation caused by thermal conduction. We will follow the terminology introduced by Foster [15] and will call it thermoconductive noise throughout this paper. Glenn first showed numerically that the thermoconductive phase noise can exceed shot noise over a wide frequency range [8]. Wanser later proposed an analytical form of the thermoconductive phase noise in passive fibers [9]. Despite being presented without a derivation, the Wanser formula has found very good agreement with experiments in several reports based on Mach-Zehnder interferometers (MZIs) [2,5–7]. Other authors have applied it to Fabry-Perot (FP) [10] and Sagnac interferometers [10–12], and experiments have shown reasonable agreement in a fiber Sagnac interferometer [3] and a fiber-optic gyro [4]. Meanwhile, a similar thermoconductive model has also been developed in the context of distributed-feedback (DFB) fiber lasers by Foster *et al.* [13] in an effort to clarify some of the theoretical questions surrounding the Wanser theory. The spectral density of the temperature fluctuations in a fiber cavity is derived by solving the Langevin equation under a fluctuational force. The model gives a slightly different spectral shape compared to the Wanser theory but shows excellent agreement with experiments above 1 kHz in both fiber lasers [13] and passive fibers [7].

A limitation of the Wanser and the Foster theories is their asymptotic behavior at the low-frequency end. Whereas both theories predict a nearly frequency-independent noise spectrum below 1 kHz, data measured with both passive fibers [7] and fiber lasers [16,17] have consistently shown a $1/f$ dependence. Although there is evidence that the $1/f$ frequency noise in DFB fiber lasers could be due to nonequilibrium thermal fluctuations of the gain medium [18], the fact that the $1/f$ behavior is observed in passive fibers indicates the existence of a different type of thermal fluctuation solely associated with fibers. Recently, thermomechanical fluctuations of fiber length have been proposed as a possible source of fiber thermal noise [14]. Using the fluctuation-dissipation theorem (FDT) [19], it can be shown that such length fluctuations have a $1/f$ spectral density, which can be directly linked to phase noise in fiber interferometers or frequency noise in fiber lasers. The recent experimental study by Bartolo *et al.* has shown that combining this thermomechanical model and the Wanser (or Foster) model leads to a theoretical thermal phase noise spectrum that matches the experimental data from an MZI all the way down to 30 Hz where residual intensity noise begins to dominate [7]. A sensing noise floor reaching even lower frequencies (infrasonic frequencies 10 mHz to 30 Hz) has been probed by Gagliardi *et al.* in fiber Bragg grating FP cavities [20]. However, proper attention should be given to the ongoing debate over the source of the low-frequency noise in this paper [21,22].

Despite the recent progress, several theoretical questions remain to be answered. First, one would question if there is any underlying connection between the thermoconductive noise and the thermomechanical noise. The two types of noises have been analyzed using two completely different approaches. Their relations are not obvious based on the current theories. Therefore, a unified theory of fiber thermal noises not only offers an alternative perspective to the current theories, but also is a necessary step toward understanding the interrelations

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between the different noise mechanisms. Second, despite small differences, the two variations of the thermoconductive model from Wanser and Foster *et al.*, nevertheless, raise a question in practice: Which one should we use? The Wanser theory is more generic in the sense that it was developed based on passive fibers. However, the lack of a derivation undermines its utility for further discussions. The Foster theory, on the other hand, was derived for DFB fiber lasers. Although the result is not restricted to active systems or FP configurations, it is, nonetheless, desirable to look for a theory completely based on passive fibers. Finally, the current theory for the thermomechanical noise is only valid for frequencies much lower than the first resonance of the longitudinal vibration [14]. It is unclear how the noise behaves beyond this limit. For fibers longer than a couple of meters, this resonant frequency falls below 1 kHz. Therefore, it is necessary to search for an alternative approach, which can extend the theory to a wider frequency range.

In the present paper, we attempt to bring both types of thermal noises under a universal framework based on the FDT. Along the way, we seek to address the questions mentioned above. The paper is organized as follows. In Sec. II, we will give a brief overview of the various types of fluctuations involved and how they are related to the FDT. In Sec. III, we will offer a new derivation of the thermoconductive phase noise via the FDT and will compare it with the Wanser and the Foster theories. In Sec. IV, we will reinvestigate the thermomechanical noise using a generalized scheme, which may shed some light on the behavior of the noise beyond the first mechanical resonance. Finally, in Sec. V, we will give a theoretical prediction of the overall thermal phase noise and will discuss the limitations of the current method.

II. A PHYSICAL OVERVIEW

Based on all the evidence available so far, three spontaneous processes can cause the phase of the light traveling through a fiber to fluctuate. They are as follows: (a) spontaneous thermal expansion (STE), i.e. random extension and contraction of the fiber caused by spontaneous fluctuations of the local temperatures through a nonzero thermal expansion coefficient, (b) spontaneous thermo-optic (STO) (or thermorefractive) effect, i.e., random variations in the refractive index induced by spontaneous fluctuations of the local temperatures, and (c) spontaneous expansion originating from mechanical effects, i.e., random extension and contraction of the fiber due to internal friction (or structural damping).

Among these three processes, STE and STO both originate from the random fluctuations of local temperatures. Even in a global thermodynamic equilibrium, the temperature in any real object exhibits local fluctuations. These fluctuations cause extension and contraction of the material as well as changes in refractive index. Because of their common origins, STE and STO can be combined into a single parameter when relating phase noise to temperature fluctuations [13],

$$S_\varphi(\omega) = \frac{4\pi^2 l^2}{\lambda^2}\left(\frac{dn}{dT} + n\alpha_L\right)^2 S_{\delta T}(\omega), \tag{1}$$

where $S_{\delta T}(\omega)$ and $S_\varphi(\omega)$ are the power spectral densities of the local temperature fluctuations and the thermoconductive phase noise, respectively, ω is the angular frequency, λ is the wavelength in vacuum, l is the nominal length of the fiber, n is the refractive index, α_L is the linear expansion coefficient, and T is temperature. The two terms inside the parentheses correspond to STO and STE, respectively. The sum of them leads to a simple proportional relation between the thermoconductive phase noise and the local temperature fluctuations. According to the classic theory of elasticity, structural deformation in a solid body is coupled to local temperature deviation through thermoelasticity [23]. This coupling causes thermoelastic damping, which is characterized by a frequency-dependent loss $\phi = \Delta\omega\,\tau/(1+\omega^2\tau^2)$, where ϕ is the loss angle in the generalized Hooke's law, Δ is the relaxation strength, and τ is the relaxation time [23]. STE and thermoelasticity are indeed two different views of the same microscopic process. They are fundamentally linked by the FDT. This connection has been elegantly demonstrated by Braginsky *et al.* in the study of the mirror noise for the Laser Interferometer Gravitational-Wave Observatory (LIGO) [24].

Experimentally, it has been shown that there is a second type of internal damping widely existing in solid materials, which shows weak-frequency dependence over broad spectral ranges [25–27]. It is commonly called internal friction, or sometimes, structural damping. Internal friction in a crystalline solid has been attributed to thermally activated defect motion [28]. The microscopic picture of internal friction in a noncrystalline material, such as glass is not as clear, except being generally associated with the Brownian motion [29]. However, the FDT predicts that there exists a type of spontaneous structural fluctuation (under the equilibrium condition) directly linked to this damping mechanism. This is what we refer to as the thermomechanical noise, which, so far, can only be analyzed through the FDT.

From the discussion above, it is evident that the FDT can provide a universal framework under which all the random processes involved for fiber thermal noises can be treated in a similar fashion. In the following, we will outline the general steps for deriving a thermal noise spectral density using the FDT.

The general expression for the FDT, according to Callen and Greene [19], can be written as

$$S_\xi(\omega) = \frac{k_B T}{\pi\omega^2}\mathrm{Re}[Y_\xi(\omega)], \tag{2}$$

where $S_\xi(\omega)$ is the spectral density of the spontaneous fluctuations of an extensive parameter ξ of the system, $Y_\xi(\omega)$ is the admittance function associated with ξ, and k_B is the Boltzmann constant. The FDT allows one to find $S_\xi(\omega)$ through the real part of $Y_\xi(\omega)$, which is the loss of the system under a harmonic external excitation at the frequency ω. This external excitation, denoted as $\tilde{F}(\omega)$ here, is a Fourier component of a generalized force $F(t)$, which forms a Hamiltonian with ξ as $H_\xi = F(t)\xi$. The admittance function is related to $\tilde{F}(\omega)$ and ξ through

$$Y_\xi(\omega) = \left(\frac{\tilde{F}(\omega)}{i\omega\tilde{\xi}(\omega)}\right)^{-1}, \tag{3}$$

where $\tilde{\xi}(\omega)$ denotes the Fourier component of ξ at the frequency ω under the excitation of $\tilde{F}(\omega)$. As a simple example, we consider a mechanical oscillator with a spring constant satisfying the generalized Hooke's law, $F_r = -k[1 + i\phi(\omega)]x$, where F_r is the restoring force, k is the ideal spring constant, $\phi(\omega)$ is the loss angle, and x is displacement. It is straightforward to show from (3) that

$$Y_x(\omega) = \frac{\omega k\phi(\omega) + i(\omega k - m\omega^3)}{(k - m\omega^2)^2 + k^2\phi^2(\omega)}, \tag{4}$$

where m is mass. Substituting (4) into (2), the spectral density of spontaneous displacement fluctuation can be written as

$$S_x(\omega) = \frac{k_B T \omega_0^2 \phi(\omega)}{\pi\omega m\left[\left(\omega^2 - \omega_0^2\right)^2 + \omega_0^4\phi^2(\omega)\right]}, \tag{5}$$

where $\omega_0^2 \equiv k/m$ is the angular resonant frequency of the oscillator.

Through the above discussion, the following steps must be taken when the FDT is used to analyze the spontaneous fluctuations of a general variable ξ: (i) identify the generalized force F corresponding to ξ, (ii) determine the admittance function $Y_\xi(\omega)$ and its real part, and (iii) find the spectral density of ξ directly through the FDT.

For the aforementioned three spontaneous processes in fibers, STE and STO are both related to temperature fluctuations as shown by (1). So, they can be treated together by finding the temperature spectral density $S_{\delta T}(\omega)$. The generalized force in this case must be in the unit of entropy. Note that STE can also be treated separately with the fiber length as the fluctuating variable. But we have chosen not to pursue this route for the obvious reason of combining STE and STO in one single formula. Meanwhile, fiber length is the variable in concern for the thermomechanical noise. The generalized force, hence, takes the unit of actual force.

For continuous bodies, such as fibers, the main challenge of using the FDT is how to determine $\mathrm{Re}[Y_\xi(\omega)]$. A conceptually simple strategy is to decompose the fluctuation of ξ into the superposition of basis functions or "normal modes" with each normal mode representing a solution of the eigenvalue problem for ξ [30]. This mode-expansion approach is particularly suitable for one-dimensional (1D) problems involving mechanical vibrations because each normal mode can be treated as a simple oscillator whose admittance function takes the form of (4). The overall spectral density is simply the sum of the spectral densities of the individual normal modes. We will use this approach to analyze the thermomechanical noise. One important advantage of mode expansion, when compared with the direct method used in Ref. [14], is that it does not require the frequency to be well below the first mechanical resonance. This allows the theory to be extended to a wider frequency range, as is shown in Sec. IV. As for the thermoconductive noises, finding the admittance function for the temperature fluctuation is a much more challenging task. Levin has outlined a scheme that relates the real part of the admittance function to the dissipated energy under external harmonic driving [31,32]. In Sec. III, we will solve for $S_{\delta T}(\omega)$ using this method.

Before moving on to specific discussions, it should be stressed here that, throughout this article, we use the following Fourier transform convention:

$$f(t) = \int_{-\infty}^{\infty} g(\omega)e^{-i\omega t}d\omega \quad \text{and} \quad g(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)e^{i\omega t}dt. \tag{6}$$

In addition, all power spectral densities are two sided. These choices of conventions account for the 4π difference in the expression of the FDT, i.e., Eq. (2), from some of the references, for example, Refs. [14,29–32]. Ultimately, the scale of the thermal noise spectra critically depends on the conventions used to derive them. Unfortunately, these conventions have not been explicitly made clear in some of the earlier papers on fiber thermal noise. This could contribute to some of the confusion and inconsistencies in the comparisons between theories and experiments.

III. THE THERMOCONDUCTIVE NOISE

In this section, we intend to use the FDT to derive the spectral density of local temperature fluctuations in a single-mode fiber. To that end, we will follow the thought experiment proposed by Levin [31,32] for the analysis of the LIGO mirror noise. In order to calculate the energy dissipation of the system, we imagine introducing a harmonic perturbation with the general form of $F_0\cos(\omega t)f(\vec{r})$, where F_0 is a proportional factor characterizing the amplitude of the external force and $f(\vec{r})$ is a form factor corresponding to the spatial distribution of the perturbation. We then calculate the power dissipation of the system under the external driving. This power dissipation W_{diss} is related to the admittance function by $|\mathrm{Re}[Y_\xi(\omega)]| = 2W_{\mathrm{diss}}/F_0^2$. Thus, the FDT can be written as

$$S_\xi(\omega) = \frac{2k_B T}{\pi\omega^2}\frac{W_{\mathrm{diss}}}{F_0^2}. \tag{7}$$

Now, let us consider a section of single-mode fiber with a laser beam propagating through it. The thermodynamic temperature fluctuations in the fiber imprint a phase noise on the transmitted light, which can be probed by a phase detector at the end of the fiber. As pointed out in Sec. II, the perturbing force for temperature fluctuations must be in the unit of entropy. It is also conceivable that this entropy perturbation must be injected throughout the beam path with the form factor $f(\vec{r})$ matching the laser beam profile. This is because the laser beam samples local fluctuations of the fiber throughout its path, and the detected phase noise is weighted by the Gaussian power profile of the fiber mode as illustrated in Fig. 1.

For the purpose of analysis, we assume the fiber is a straight cylindrical glass rod extending along the z axis of a cylindrical coordinate. The assumption of straight fiber is not absolutely necessary though because, as long as the fiber is not sharply

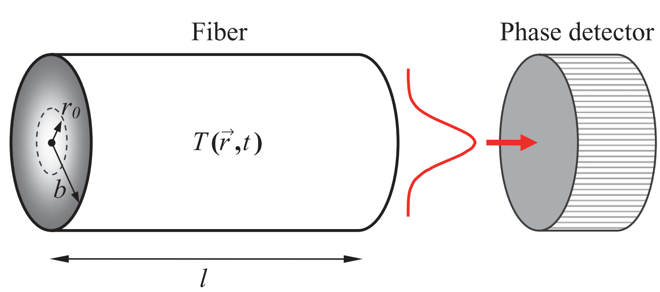

FIG. 1. (Color online) A conceptual model for the thermal phase noises imprinted on a laser beam propagating through a section of passive fiber due to spontaneous fluctuations in the fiber.

bent, the laser mode can be treated as symmetrical around the optical axis and uniform along the fiber. As we will see later, these are the only requirements needed for the derivation. The volumetric density of the injected entropy at any moment t and any radial location r from the optical axis can be expressed as

$$\delta S(r,t) = \frac{F_0}{\pi r_0^2 l} e^{-r^2/r_0^2} e^{-i\omega t}, \tag{8}$$

where r_0 is effective radius of the Gaussian power profile and l is the fiber length as shown in Fig. 1. Note that the denominator is due to the normalization of the form factor as $\int f(\vec{r}) d^3 r = 1$. The entropy injection creates heat, which changes the temperature field inside the fiber through thermal conduction. Assuming the fiber is in thermal equilibrium at temperature T without the perturbation, following the general theory of thermal conduction [33], the equation for temperature variations can be written as

$$\kappa \nabla^2 \delta T - C_V \frac{\partial\, \delta T}{\partial t} = T \frac{\partial\, \delta S}{\partial t}, \tag{9}$$

where $\delta T(\vec{r},t)$ is the variation in the temperature field induced by the perturbation, κ is the thermal conductivity, and C_V is the volumetric heat capacity. Since the entropy injection only depends on r, we can substitute (8) into (9) to yield

$$\frac{\partial^2 \delta T}{\partial r^2} + \frac{1}{r}\frac{\partial\, \delta T}{\partial r} - \frac{1}{D}\frac{\partial\, \delta T}{\partial t} = -\frac{i\omega F_0 T}{\pi r_0^2 l \kappa} e^{-r^2/r_0^2} e^{-i\omega t}, \tag{10}$$

where $D \equiv \kappa / C_V$ is diffusivity. Equation (10) is a nonhomogeneous heat conduction equation with a harmonic source characterized by the term on the right-hand side, and the heat flow only has the radial component. Such an equation can be solved using the Green's function method [34]. The solution for $\delta T(r,t)$ is found to be (see details in the Appendix)

$$\delta T(r,t) = \frac{i\omega F_0 T}{4\pi l \kappa} e^{-i(\omega t + \psi_0)} \int_1^\infty \exp\left(i\psi_0 \zeta - \frac{r^2}{r_0^2 \zeta} \right) \frac{d\zeta}{\zeta}, \tag{11}$$

where $\psi_0 \equiv \omega r_0^2/(4D)$ and ζ is defined by (A7). The temperature differences along the radial direction break the thermal equilibrium. As the system tries to "relax" back to thermal equilibrium, heat flow is generated along the direction of $-\nabla\, \delta T = -\partial\, \delta T/\partial r$. This relaxation process via thermal conduction causes energy dissipation, which, according to the theory of thermal conduction [35], can be expressed as

$$W_{\text{diss}} = \int \frac{\kappa}{T} \langle (\nabla\, \delta T)^2 \rangle dV = \frac{\kappa}{2T} \int \nabla\, \delta T \cdot \nabla\, \delta T^* dV, \tag{12}$$

where the $*$ represents the complex conjugate and the volumetric integration is over the entire fiber. By substituting (11) into (12) and introducing an independent variable of integration ζ' for $\nabla\, \delta T^*$, the power dissipation due to thermal conduction can be written as

$$W_{\text{diss}} = \frac{\omega^2 F_0^2 T}{4\pi r_0^4 l \kappa} \int_1^\infty \int_1^\infty V(\zeta,\zeta') \frac{e^{i\psi_0(\zeta-\zeta')}}{\zeta^2 \zeta'^2} d\zeta\, d\zeta', \tag{13}$$

with

$$V(\zeta,\zeta') = \int_0^b \exp\left[-\frac{r^2}{r_0^2}\left(\frac{1}{\zeta} + \frac{1}{\zeta'} \right) \right] r^3 dr, \tag{14}$$

where b is the outer radius of the fiber as shown in Fig. 1. A particularly simple expression of $V(\zeta,\zeta')$ is obtained when infinite cladding diameter, i.e., $b \to \infty$, is assumed. In this case, we find $V(\zeta,\zeta') = r_0^4/[2(1/\zeta + 1/\zeta')^2]$. Then, (13) can be simplified to

$$W_{\text{diss}} = \frac{\omega^2 F_0^2 T}{8\pi l \kappa} \int_1^\infty \int_1^\infty \frac{e^{i\psi_0(\zeta-\zeta')}}{(\zeta+\zeta')^2} d\zeta\, d\zeta'. \tag{15}$$

The double integral can be conveniently evaluated with a change in variables as follows:

$$\eta = \frac{\zeta+\zeta'}{2} \quad \text{and} \quad \eta' = \frac{\zeta-\zeta'}{2}. \tag{16}$$

The integrals to ζ and ζ' are then converted to integrals to η and η',

$$\int_\eta \int_{\eta'} \frac{e^{2i\psi_0\eta'}}{4\eta^2} |J|\, d\eta\, d\eta' = \int_1^\infty \frac{d\eta}{2\eta^2} \int_{-(\eta-1)}^{\eta-1} e^{2i\psi_0\eta'} d\eta', \tag{17}$$

where $|J|$ is the absolute value of the Jacobian, which, in this case, is equal to 2. Now, it is straightforward to work out the integrals. Substituting the result into (15) leads to the final form of W_{diss},

$$W_{\text{diss}} = \frac{\omega^2 F_0^2 T}{8\pi l \kappa} \mathrm{Re}[e^{2i\psi_0} E_1(2i\psi_0)], \tag{18}$$

where $E_1(x)$ is the special function of the exponential integral [36]. By substituting (18) into (7), the spectral density of the spontaneous temperature fluctuations in the fiber is found to be

$$S_{\delta T}(\omega) = \frac{k_B T^2}{4\pi^2 l \kappa} \mathrm{Re}\left[\exp\left(\frac{i\omega r_0^2}{2D} \right) E_1\left(\frac{i\omega r_0^2}{2D} \right) \right]. \tag{19}$$

The actual phase noise acquired by the laser beam can be obtained by substituting (19) into (1).

It is interesting to compare the above result with the results obtained by Wanser [9] and Foster *et al.* [13]. Note that the fiber mode here is characterized by its power profile radius r_0, which is related to the mode field radius a used by the other authors through the relation $r_0^2 = a^2/2$. Also, rather than assuming a fiber of unit length as in the Foster theory, here, fiber length is kept explicit throughout the analysis. With these considerations, it is immediately obvious that (19) is identical to the spectral density given by Foster *et al.* (Eq. (27) in Ref. [13]). It should also be noted here that the same Fourier-transform convention has been used in Ref. [13] as evident from Eq. (20) of the paper.

The fact that the Foster theory, which was initially derived using the conventional statistical approach, can be reproduced via the FDT demonstrates the feasibility of the FDT as a universal framework for analyzing different thermal noises in optical fibers. The analysis given above confirms that the Foster theory is also valid for passive fibers. Moreover, it manifests a deeper relation between the thermoconductive noise and the thermomechanical noise. From the FDT point of view, they simply correspond to two different channels of internal dissipation. For the thermoconductive noise, it is the temperature relaxation through thermal conduction, and for the thermomechanical noise, it is the internal friction caused by Brownian motion.

IV. THE THERMOMECHANICAL NOISE

The spontaneous reshaping of a continuous solid body due to mechanical damping can, in principle, be analyzed via normal mode expansion [30]. Each normal mode can be treated as a harmonic oscillator whose fluctuation spectral density is given by (5). The total spectral density is the sum of the contributions from all the individual modes as expressed by

$$S_\xi(\omega) = \frac{k_B T}{\pi\omega}\sum_N \frac{\omega_N^2\phi_N(\omega)}{m_N\left[\left(\omega^2-\omega_N^2\right)^2+\omega_N^4\phi_N^2(\omega)\right]}, \tag{20}$$

where ω_N, m_N, and $\phi_N(\omega)$ are the angular frequency, effective mass, and loss angle associated with the Nth mode, respectively. For length fluctuations, the fiber can be further simplified to a 1D system so that only the longitudinal vibration modes are involved. The validity of the 1D assumption has been previously demonstrated at low frequencies [14]. Further discussions about the limitations of this model will be deferred to the next section. If the fiber is treated as a long thin rod along the z axis as illustrated in Fig. 1, the longitudinal elastic wave equation can be written as [33]

$$\frac{\partial^2 u_z}{\partial z^2} - \frac{\rho}{E_0}\frac{\partial^2 u_z}{\partial t^2} = 0, \tag{21}$$

where u_z is longitudinal displacement, ρ is the density, and E_0 is the Young's modulus without loss. Under the boundary conditions of two free ends at $z=0$ and $z=l$, the Nth vibration mode is given by $u_{zN}(z,t) = w_N(z)e^{-i\omega_N t}$, where $w_N(z) = \cos(N\pi z/l)$ and $\omega_N = (N\pi/l)\sqrt{E_0/\rho}$. The effective mass for the Nth mode can be obtained using the formula [30],

$$m_N = \int \rho(\vec{r})w_N^2(\vec{r})dV = \rho A\int_0^l w_N^2(z)dz, \tag{22}$$

where A is the cross-sectional area of the fiber. It immediately becomes clear that $m_N = \rho A l/2$. Substituting ω_N and m_N into (20) yields the general expression for the spectral density of fiber length fluctuations,

$$S_l(\omega) = \frac{2k_B T l}{\pi^3 A E_0\omega}\sum_N \frac{\phi_N(\omega)}{N^2\left[\left(1-\omega^2/\omega_N^2\right)^2+\phi_N^2(\omega)\right]}. \tag{23}$$

For the special case of structural damping, if we treat the loss angle as approximately frequency independent over the entire spectral range of interest [29], the spectral density is simplified to

$$S_l(\omega) = \frac{2k_B T l\phi_0}{\pi^3 A E_0\omega}\sum_N \frac{1}{N^2\left[\left(1-\omega^2/\omega_N^2\right)^2+\phi_0^2\right]}. \tag{24}$$

At the low-frequency end, when the frequency is well below the first normal mode frequency, i.e., $\omega^2/\omega_1^2 \ll 1$, the summation in (24) reduces to $\sum_N 1/[N^2(1+\phi_0^2)]$. It can be further simplified to $\sum_{N=1}^{\infty} 1/N^2 = \pi^2/6$ since $\phi_0 \ll 1$ for glass [25–27]. Then, the spectral density becomes

$$S_l(\omega) = \frac{k_B T l\phi_0}{3\pi A E_0\omega}. \tag{25}$$

Taking into account the factor 4π caused by the use of different conventions as pointed out in Sec. II, (25) is consistent with the previously reported result, i.e., Eq. (4) in Ref. [14].

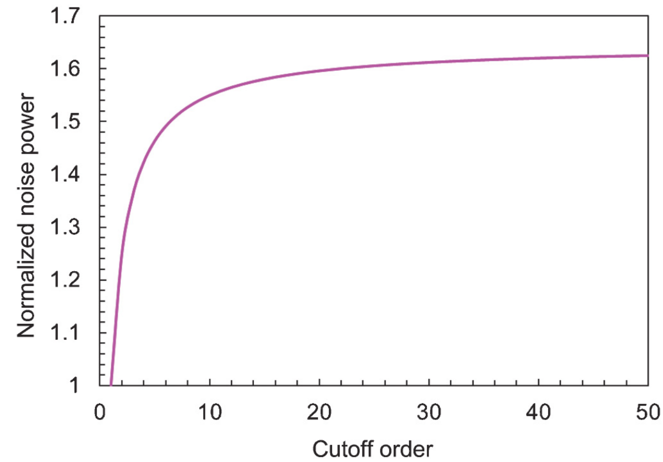

FIG. 2. (Color online) The dependence of the normalized noise power, which is defined as the total noise power over the power of the first normal mode on the cutoff order. $\phi_0 = 0.01$ is assumed in the computation.

When the frequency is near or above the first normal mode frequency, (24) must be used to compute the frequency dependence of the spectral density. This, in general, has to be performed numerically. The computation of $S_l(\omega)$ requires a cutoff order of the normal modes to set an upper limit on the summation. Several aspects need to be considered when evaluating the impact of higher-order modes. One logical criterion is the contribution of each order to the overall noise power. This can be performed by integrating each order in (24) over the entire frequency span. Figure 2 shows how the total noise power, normalized to the power of the first order, changes against the cutoff order. It is evident that further increasing the total number of orders beyond $N = 10$ makes only a small difference in the total noise power. This simple power consideration, however, is not sufficient to characterize the impact of each individual mode. As (24) shows, each mode can dominate the noise spectrum near its resonance over the total contribution from all other modes even when its share in the total noise power is negligible. For example, the spectral density at the qth resonant frequency can be written as

$$S_l(\omega_q) = \frac{2k_B T l\phi_0}{\pi^3 A E_0\omega_1}\left\{\frac{1}{q^3\phi_0^2} + \sum_{N\neq q}\frac{1}{qN^2\left[(1-q^2/N^2)^2+\phi_0^2\right]}\right\}, \tag{26}$$

with the first term in the curly braces representing the peak value of the qth normal mode and the second term representing the sum of the rest of the modes. Taking $\phi_0 = 0.01$ and $q = 10$, the first term is roughly a factor of 120 greater than the second term. However, a close look at (26) reveals that the peak values at the resonances decline at $1/q^3$, whereas the nonresonance floor declines at approximately $1/q$ (see the dashed curves in Fig. 3). As a result, the dominance of an individual normal mode at its resonance will eventually cease as q increases. This upper limit of the normal mode order can be found by equating the two terms in the curly braces in (26). For $\phi_0 = 0.01$, this leads to $q \approx 100$. Practically, when only the

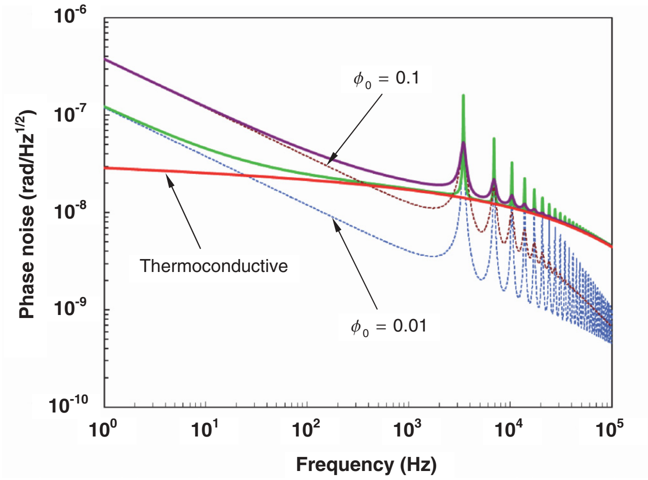

FIG. 3. (Color online) The thermal phase noise spectra for red solid line: thermoconductive noise; blue dotted line: thermomechanical noise with $\phi_0 = 0.01$; and brown dotted line: $\phi_0 = 0.1$ as well as the corresponding total phase noise for green solid line: $\phi_0 = 0.01$ and purple solid line: $\phi_0 = 0.1$.

total noise is of concern, the higher-order mechanical modes may not play a significant role in the total noise spectrum due to the dominance of the thermoconductive noise at high frequencies as shown in Fig. 3. Thus, they can be dropped from the calculation with little impact to the total noise spectrum.

V. DISCUSSIONS AND CONCLUSIONS

The overall thermal phase noise imprinted on the transmitted laser beam is equal to the sum of the contributions from the thermoconductive noise and the thermomechanical noise, i.e.,

$$S_\varphi(\omega) = \frac{4\pi^2}{\lambda^2}\left[\left(\frac{dn}{dT} + n\alpha_L\right)^2 l^2 S_{\delta T}(\omega) + n^2 S_l(\omega)\right], \quad (27)$$

where $S_{\delta T}(\omega)$ and $S_l(\omega)$ are given by (19) and (24), respectively. Taking the example of a 1-m-long single-mode fiber operating at 1550 nm under a temperature of 298 K, the following optical and thermal properties of silica are used [7]: $n = 1.457$, $dn/dT = 9.488 \times 10^{-6}\mathrm{K}^{-1}$, $a = 2.605\ \mu\mathrm{m}$, $\alpha_L = 5.0 \times 10^{-7}\mathrm{K}^{-1}$, $\kappa = 1.37\ \mathrm{W\ m^{-1}\ K^{-1}}$, and $D = 0.82 \times 10^{-6}\mathrm{m^2/s}$. The mechanical properties are somewhat difficult to identify since an actual optical fiber is a compound mechanical system consisting of a silica central core surrounded by a concentric acrylate buffer. Experimental results on some of the basic mechanical parameters of fibers are not always consistent [27,37]. Here, we choose the fiber outer radius of $b = 125\ \mu\mathrm{m}$, an averaged density of $\rho = 1430\ \mathrm{kg/m^3}$ [27], and a measured Young's modulus of $E_0 \approx 68$ GPa [37]. Figure 3 shows the calculated phase noise spectra of the thermoconductive noise, the thermomechanical noise, and the total noise. A factor of $\sqrt{2}$ has been multiplied in the calculation to convert the two-sided noise spectra into one-sided noise spectra. The thermomechanical noise has been calculated using two different loss angles $\phi_0 = 0.01$ and $\phi_0 = 0.1$ with ϕ_0 kept the same across the entire frequency span in each case. For both values of ϕ_0, the thermoconductive noise dominates the total noise at high frequencies (e.g., >1 kHz) whereas the thermomechanical noise prevails at low frequencies (e.g., <100 Hz). Such a general behavior agrees well with experimental observations.

The resonance peaks of the normal modes in the thermomechanical noise spectra cause similar spikes in the total noise spectra in Fig. 3. Such features, however, have not been observed experimentally so far. This indicates possible oversimplifications in the model for the thermomechanical noise. One obvious limitation of the current model lies in the assumption of a constant ϕ_0 over several decades of frequencies. Due to the lack of specific experimental results about ϕ_0 in this frequency range, we are forced to assume $\phi_0 = 0.01$ based on a measurement performed at 75–200 kHz [27]. Several factors may weaken or even may invalidate this assumption. For example, measurements made with uncoated fused-silica fibers have shown a thermoelastic loss peak at roughly 10–1000 Hz [26]. Also, the acrylate polymer coating of optical fibers may exhibit strong viscoelasticity characterized by frequency-dependent loss angle and modulus [38]. Another aspect under scrutiny is the limitation of the 1D model. By using (21)–(23), we have assumed that only longitudinal displacements are involved in the spontaneous extension and contraction of the fiber. A more refined model would have to treat a fiber as a three-dimensional body and would have to consider the lateral motions induced by the longitudinal displacements. In this case, the inertia of the lateral motion must be considered, and the equation of motion is no longer (21) [39]. A similar aspect worth considering is the excitation of the bending modes. The bending modes do not contribute to the laser phase noise, but they provide extra dissipation under external excitation. All these above effects could potentially raise the loss angle over the entire, or a portion of the, interested frequency span. Higher loss angles would suppress the resonance peaks as demonstrated by the $\phi_0 = 0.1$ trace in Fig. 3, and such resonance features could be too small to be detected under realistic conditions. A complete thermomechanical model would require a three-dimensional treatment of a clad rod, which is out of the scope of the present paper. Despite these limitations, 1D mode expansion is able to offer a picture of the thermomechanical noise beyond very low frequencies. This is, nonetheless, one step forward from the previous model. It should also be pointed out that there have been very few experiments that report the observation of the $1/f$ noise in passive fibers so far [7,20], and some of them are still up for debate [20–22]. Ultimately, verification of the theoretical predictions in this frequency range likely requires experiments specifically designed to answer these questions.

In conclusion, a unified general treatment of the spontaneous thermal fluctuations in optical fibers has been presented based on the FDT. Two types of thermal noises have been analyzed. For the thermoconductive noise, the spectral density of the spontaneous temperature fluctuation has been derived for passive fibers using the FDT. An identical result has been reached as the Foster theory [13], which was developed using the statistical method for fiber lasers. The method presented here not only provides an alternative view of the thermoconductive noise, but also offers an independent confirmation to the Foster theory as well. For the thermomechanical noise, a new scheme combining normal-mode expansion and the

FDT is able to extend the discussion of this noise mechanism to beyond very low frequencies, although a more refined model is probably needed to explain the noise behavior near mechanical resonances. Besides offering independent confirmations for the existing theories, the general treatment, based on the FDT, highlights the underlying relation between the thermoconductive noise and the thermomechanical noise, i.e., they are simply the manifestation of two different channels of energy dissipation. In addition, the importance of the Fourier-transform conventions to the comparison of noise spectra from different theories and experiments has been pointed out and has been demonstrated.

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APPENDIX: USING GREEN'S FUNCTION TO SOLVE RADIAL HEAT FLOW IN A FIBER WITH A GAUSSIAN HARMONIC SOURCE

The partial differential equation (10) can be solved in closed form under the assumption that the fiber can be treated as an infinitely large cylinder. This is generally a valid assumption as the fiber-mode radius is approximately 5 μm for single-mode fibers at the 1.5-μm wavelength region and the outer diameter of the cladding is usually 125 μm. The Green's function for radial heat flow in a cylindrical thermal conductor with an infinite radius is [34]

$$G(r,t|r',t') = \frac{1}{4\pi D(t-t')}\exp\left[-\frac{r^2+r'^2}{4D(t-t')}\right] \times I_0\left[\frac{rr'}{2D(t-t')}\right], \tag{A1}$$

where $I_0(x)$ is the zeroth-order modified Bessel function of the first kind. The solution of (10) can be directly expressed by the following integral:

$$\delta T(r,t) = 2\pi D\int_{-\infty}^{t}\int_0^\infty G(r,t|r',t')g(r',t')r'dr'dt', \tag{A2}$$

where $g(r',t')$ is the external excitation in (10), i.e.,

$$g(r',t') = \frac{i\omega F_0 T}{\pi r_0^2 l\kappa}e^{-r'^2/r_0^2}e^{-i\omega t'}. \tag{A3}$$

Note that the integration range for t' has been changed from the normal (0, t) to ($-\infty$, t) due to the periodic nature of the integrant. Substituting (A1) and (A3) into (A2) yields

$$\delta T(r,t) = \frac{i\omega F_0 T}{2\pi r_0^2 l\kappa}e^{-i\omega t}\int_0^\infty \frac{h(r,\tau)}{\tau}\exp\left(-\frac{r^2}{4D\tau}+i\omega\tau\right)d\tau, \tag{A4}$$

where $\tau \equiv t - t'$ and $h(r,\tau)$ represents the integral,

$$h(r,\tau) = \int_0^\infty J_0(\alpha r')e^{-p^2r'^2}r'dr', \tag{A5}$$

with $\alpha \equiv ir/(2D\tau)$, $p^2 \equiv 1/r_0^2 + 1/(4D\tau)$, and $J_0(x)$ denoting the zeroth-order Bessel function of the first kind. The integral of $h(r,\tau)$ can be evaluated with the help of the Weber's first exponential integral [40], yielding an analytical expression,

$$h(r,\tau) = \frac{2D\tau}{\zeta}\exp\left(\frac{r^2}{4D\tau\zeta}\right), \tag{A6}$$

where

$$\zeta \equiv 1 + 4D\tau/r_0^2. \tag{A7}$$

Substituting (A6) into (A4) and changing the variable of the integral from τ to ζ lead to formula (11).

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