

A METHOD TO IMPROVE THE SENSITIVITY OF RADIO TELESCOPES

RICHARD LIEU¹, T.W. B. KIBBLE², AND LINGZE DUAN¹

¹ Department of Physics, University of Alabama, Huntsville, AL 35899, USA

² Blackett Laboratory, Imperial College, London SW7 2AZ, UK

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ABSTRACT

As an extension of the ideas of Hanbury-Brown and Twiss, a method is proposed to eliminate the phase noise of white chaotic light in the regime where it is dominant, and to measure the much smaller Poisson fluctuations from which the incoming flux can be reconstructed. The best effect is achieved when the timing resolution is finer than the inverse bandwidth of the spectral filter. There may be applications to radio astronomy at the phase noise dominated frequencies of 1–10 GHz, in terms of potentially increasing the sensitivity of telescopes by an order of magnitude.

Key words: instrumentation: detectors – radiation mechanisms: general – techniques: interferometric

1. INTRODUCTION AND BACKGROUND

In astronomical observations, depending upon the radiation bandpass of interest, usually one of the two natural components of flux uncertainties, viz., phase and Poisson fluctuations, plays the principal role of masking the genuine incoming signal of celestial sources. Specifically, while at radio frequencies of photon abundance classical phase noise dominates any estimate of the necessary exposure time (e.g., Burke & Graham-Smith 2010), at higher frequencies it is the Poisson counting statistics of individual photons that matter (e.g., Birney et al. 2006). The intention of this paper is to propose a method of eliminating the phase noise fluctuations in the wavelength region where they are dominant. While of little significance to efforts in improving sensitivities in the optical and X-ray telescopes, there may be applications to radio astronomy.

To setup the background, we consider a unidirectional beam of cross-sectional area A propagating in the z direction, and for simplicity assume that there is only one polarization mode, e.g., plane polarization in the x direction.

In a quantum treatment, the positive frequency part of the electric field operator is given (in units with $c = \hbar = 1$) by

$$\hat{E}^{(+)}(t, z) = \frac{i}{\sqrt{4\pi A}} \int d\omega \sqrt{\omega} \hat{a}(\omega) e^{-i\omega(t-z)}, \quad (1)$$

where $\hat{a}(\omega)$ and $\hat{a}^\dagger(\omega)$ are respectively the annihilation and creation operators of the radiation mode of frequency ω , while the negative frequency part is the complex conjugate,

$$\hat{E}^{(-)}(t, z) = \hat{E}^{(+)}(t, z)^\dagger. \quad (2)$$

The cycle-averaged intensity operator $\hat{I}(t, z)$, which represents the energy flux density (the magnitude of the Poynting vector) averaged over a cycle and integrated over the area A , is given by

$$\hat{I}(t, z) = 2A \hat{E}^{(-)}(t, z) \hat{E}^{(+)}(t, z). \quad (3)$$

In the case of a stationary light beam, there is no correlation between different frequencies, and we may write

$$\langle \hat{a}^\dagger(\omega) \hat{a}(\omega') \rangle = n(\omega) \delta(\omega - \omega'), \quad (4)$$

where the angle brackets denote the ensemble averaged expectation value. Thus

$$\bar{I} \equiv \langle \hat{I}(t, z) \rangle = \int \frac{d\omega}{2\pi} \omega n(\omega), \quad (5)$$

where $n(\omega)$ is the photon occupation number of mode ω . For example, if we have a Gaussian wave form with central frequency ω_0 and reciprocal bandwidth τ , viz.,

$$n(\omega) = \sqrt{2\pi} n_0 e^{-(\omega - \omega_0)^2 \tau^2 / 2}, \quad (6)$$

where

$$\tau = \frac{1}{\delta\omega} = \frac{1}{2\pi \delta\nu}, \quad (7)$$

then

$$\bar{I} = \frac{\omega_0 n_0}{\tau}, \quad (8)$$

so that the mean rate of arrival of photons is n_0/τ . Such a spectrum typically arises from broad band emission transmitted through a narrow filter.

2. ORIGIN OF THE TWO COMPONENTS OF RADIO NOISE

It is useful to begin with a recapitulation of the standard derivation of intensity fluctuation. A real measurement will always occupy a finite length of time, so let us consider the intensity I averaged over some time interval $T \gg 1/\omega_0$

$$\hat{I}_T = \frac{1}{T} \int_0^T \hat{I}(t) dt. \quad (9)$$

Obviously,

$$\langle \hat{I}_T \rangle = \langle \hat{I} \rangle = \bar{I}. \quad (10)$$

To find the variance, we consider

$$\begin{aligned} \langle \hat{I}_T^2 \rangle &= \frac{4A^2}{T^2} \int_0^T dt_1 dt_2 \langle \hat{E}^{(-)}(t_1) \hat{E}^{(+)}(t_1) \hat{E}^{(-)}(t_2) \hat{E}^{(+)}(t_2) \rangle \\ &= \frac{1}{(2\pi)^2 T^2} \int_0^T dt_1 dt_2 \int d\omega_1 d\omega'_1 d\omega_2 d\omega'_2 \sqrt{\omega_1 \omega'_1 \omega_2 \omega'_2} \\ &\quad \times e^{i(\omega_1 - \omega'_1)t_1 + i(\omega_2 - \omega'_2)t_2} \langle \hat{a}^\dagger(\omega_1) \hat{a}(\omega'_1) \hat{a}^\dagger(\omega_2) \hat{a}(\omega'_2) \rangle. \quad (11) \end{aligned}$$

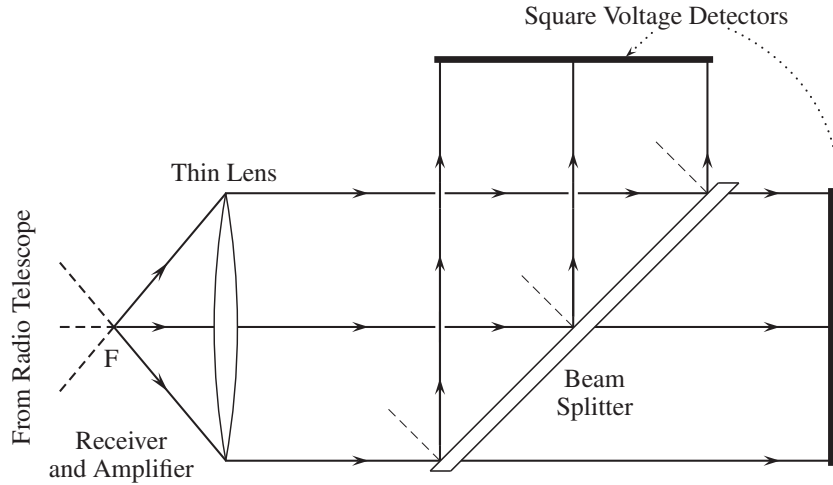


Figure 1. Radio signal from a celestial source, after it is focused by the telescope and amplified, is passed through a “Hanbury-Brown and Twiss” thin beam splitter *without* recombining. The intensity time profiles of the two beams are then measured after they traversed equal path lengths.

Now in the case of a beam with Gaussian statistics, the four-point function can be written as a sum of products of two-point functions:

$$\begin{aligned} & \langle \hat{a}^\dagger(\omega_1) \hat{a}(\omega'_1) \hat{a}^\dagger(\omega_2) \hat{a}(\omega'_2) \rangle \\ &= \langle \hat{a}^\dagger(\omega_1) \hat{a}(\omega'_1) \rangle \langle \hat{a}^\dagger(\omega_2) \hat{a}(\omega'_2) \rangle + \langle \hat{a}^\dagger(\omega_1) \hat{a}(\omega'_2) \rangle \langle \hat{a}(\omega'_1) \hat{a}^\dagger(\omega_2) \rangle \end{aligned} \quad (12)$$

Substituting the first term here into Equation (11) clearly reproduces \bar{I}^2 . The variance is therefore given by the second term. Using Equation (4), together with the commutator

$$[\hat{a}(\omega'), \hat{a}^\dagger(\omega)] = \delta(\omega' - \omega), \quad (13)$$

we find

$$\begin{aligned} \Delta I_T^2 &= \langle \hat{I}_T^2 \rangle - \langle \hat{I} \rangle^2 = \frac{1}{(2\pi)^2 T^2} \int_0^T dt_1 dt_2 \\ &\times \int d\omega_1 d\omega_2 \omega_1 \omega_2 n(\omega_1) [n(\omega_2) + 1] e^{i(\omega_1 - \omega_2)(t_1 - t_2)}. \end{aligned} \quad (14)$$

In practice we always have $\omega_0 \gg 1/\tau$, and under those circumstances the factors of ω_j can be replaced to good accuracy with ω_0 . Where the frequency profile is Gaussian, Equation (6), this gives

$$\Delta I_T^2 = \frac{\omega_0^2}{\tau T} \left[n_0^2 F\left(\frac{T}{\tau}\right) + n_0 \right], \quad (15)$$

where

$$F(x) = \sqrt{\pi} \operatorname{erf}(x) - (1 - e^{-x^2})/x. \quad (16)$$

We note the limiting forms

$$\begin{aligned} F(x) &\approx \sqrt{\pi} - 1/x, & x \gg 1, \\ F(x) &\approx x - x^3/3, & x \ll 1. \end{aligned} \quad (17)$$

The last term of Equation (15) is the shot noise term. Equivalently,

$$\left(\frac{\Delta I_T}{\bar{I}} \right)^2 = \frac{\tau}{T} \left[F\left(\frac{T}{\tau}\right) + \frac{1}{n_0} \right]. \quad (18)$$

For radio observations where $n_0 \gg 1$ by virtue of the high system temperature (i.e., large background flux), the first term

is dominant unless T/τ is extremely small. For $T \gg \tau$, one has

$$\left(\frac{\Delta I_T}{\bar{I}} \right)^2 \approx \frac{\sqrt{\pi} \tau}{T} = \frac{1}{2\sqrt{\pi} T \delta\nu}, \quad (19)$$

in accordance with the radiometer equation (Christiansen & Högbom 1985; Burke & Graham-Smith 2010).

Note that the results in this section are valid for the intensity behavior of radio waves irrespective of their origin, which can be celestial, atmospheric, scattered ground radiation, or thermal noise of the receiver’s preamplifiers.

3. PHASE NOISE CANCELLATION BY A BEAM SPLITTER

The fluctuations described in the previous section can accurately be removed by subtracting the intensity I' of a compensating beam from the primary beam, as we demonstrate below. We first assume that both beams have equal³ mean intensity, i.e.,

$$\bar{I}^t = \bar{I}^r = \frac{1}{2} \bar{I}, \quad (20)$$

because the incident beam is divided 50:50 at a beam splitter and the intensities of the two emerging beams are measured *without* combining them; see Figure 1. In what follows it is further assumed that the path lengths traversed before measurement are equal, and the thickness of the beam splitter is small with respect to the radiation wavelength (an achievable criterion at radio frequencies).

The electric fields in the two beams can be expressed in terms of creation and annihilation operators that satisfy the same commutation relations as before:

$$[\hat{a}^t(\omega), \hat{a}^{t\dagger}(\omega')] = [\hat{a}^r(\omega), \hat{a}^{r\dagger}(\omega')] = \delta(\omega - \omega'), \quad (21)$$

but of course

$$[\hat{a}^t, \hat{a}^{r\dagger}] = 0. \quad (22)$$

The field in the reflected beam may suffer a phase change relative to the incident one, but that makes no difference because the operators appear in pairs and the overall phase cancels out.

³ The assumption of a 50:50 beam splitter is actually inessential to the arguments and conclusion of this paper.

The expectation values of products are given by expressions of the same form (4), but with

$$n^{\dagger}(\omega) = n^r(\omega) = \frac{1}{2}n(\omega). \quad (23)$$

However, we also need to evaluate such mixed expressions as $\langle \hat{a}^{\dagger}(\omega)\hat{a}^r(\omega') \rangle$. Now, in the classical case, the beam can be represented as a statistical mixture, in which the field Φ_j appears with probability p_j . (Of course, the labeling index here may actually be continuous rather than discrete, but it is sufficient to consider the discrete case.) The corresponding quantum state can be taken to be a superposition of coherent states, of the form

$$\rho = \sum_j p_j |\alpha_j\rangle \langle \alpha_j|, \quad (24)$$

where $|\alpha_j\rangle$ is an eigenstate of the annihilation operators, with eigenvalue defined by Φ_j .

What then happens when the light hits the beam splitter? Classically, for an input field Φ_j , we end up with a transmitted field $\Phi_j^t = \Phi_j/\sqrt{2}$ and a reflected field Φ_j^r of the same magnitude and possibly altered phase. However, since the overall phase will always cancel out, we can ignore it and take $\Phi_j^r = \Phi_j^t$. Quantum mechanically, in place of the coherent state $|\alpha_j\rangle$ of the incident beam we have a coherent state $|\alpha_j^t, \alpha_j^r\rangle$, an eigenstate of both sets of annihilation operators, with eigenvalues defined by Φ_j^t and Φ_j^r . When we take the expectation value of the product $a^{\dagger}(\omega)\hat{a}^r(\omega')$, we can then replace each of the operators by the corresponding eigenvalues, so the result is essentially the same as for pairs of operators from the same beam. It follows therefore that as in Equation (4)

$$\langle \hat{a}^{\dagger}(\omega)\hat{a}^r(\omega') \rangle = \frac{1}{2}n(\omega)\delta(\omega - \omega'), \quad (25)$$

ignoring any overall phase which would cancel out.

For measurements of intensity averages over an interval T it is clear that their variances $(\Delta I_T^t)^2$ and $(\Delta I_T^r)^2$ are each given by exactly the same expressions as (15) above, but with n_0 replaced by $n_0/2$. Moreover, for the cross correlation, the term quadratic in n_0 is the same, but there is no linear term (i.e., no shot noise, or the last term of Equation (15)) because of Equation (22), so

$$\langle I_T^t I_T^r \rangle - \bar{I}^t \bar{I}^r = \frac{\omega_0^2 n_0^2}{4T\tau} F\left(\frac{T}{\tau}\right). \quad (26)$$

The absence of shot noise from Equation (26) is the basis of the Hanbury-Brown and Twiss effect (Hanbury-Brown & Twiss 1957), viz., the beam splitter cannot divide photons and so each is either reflected or transmitted by random 50:50 chance.

But suppose, instead of pursuing intensity interferometry, we form the difference signal $I_T^d = I_T^t - I_T^r$. Evidently its mean is zero:

$$\langle I_T^d \rangle = 0. \quad (27)$$

The variance is given by

$$(\Delta I_T^d)^2 = \langle (I_T^t - I_T^r)^2 \rangle = (\Delta I_T^t)^2 + (\Delta I_T^r)^2 - 2[\langle I_T^t I_T^r \rangle - \bar{I}^t \bar{I}^r]. \quad (28)$$

It is clear that the quadratic terms in n_0 here will cancel exactly, leaving only the linear terms, viz.,

$$\text{var}(I_T^d) \equiv (\Delta I_T^d)^2 = \frac{n_0 \omega_0^2}{T\tau}. \quad (29)$$

Thus, although the dominant phase noise component is removed when one takes the intensity difference between the beams, information about the original incident flux is *not*. The minor component of photon shot noise affects both beams independently and now becomes the dominant component. More precisely, the merit of the proposed method is that the remaining noise in the subtracted beam contains information about the photon arrival rate; from a measurement of the variance, using Equation (29), we can determine n_0 .

4. SAMPLE VARIANCE AS ESTIMATOR OF THE POPULATION MEAN

How may this approach improve the detection sensitivity of a transient or steady source? In radio astronomy the background flux is always large, so that the phase noise is the dominant reason for the masking of faint sources and shot noise is relatively minor. The sum of the measured I_t and I_r recovers the original incident intensity I_i , but that measurement is subject to the phase noise. The intensity difference, on the other hand, fluctuates about zero according to the shot noise only, and the variance of this fluctuation also contains information about the signal.

The aim is to find a way of measuring as accurately as possible the incident intensity \bar{I} , or equivalently, in view of Equation (8), the value of n_0 . If we have available a long observation time \mathcal{T} then the accuracy with which we can directly measure \bar{I} is given by Equation (19). Suppose however that instead of making this direct measurement we use the beam splitter and measure the variance of the difference signal, using a large number N of short intervals T . The mean of the difference signal is zero, but its variance depends linearly on n_0 . So the variance provides an estimator of the value of n_0 or equivalently the incident intensity \bar{I} . To know how good this estimator is, we need to compute the variance of the variance. If N is large, this is given by

$$\text{var}[(\Delta I_T^d)^2] \approx \frac{\mu_4 - \mu_2^2}{N}, \quad (30)$$

where μ_4 is the fourth central moment of the distribution of I_T^d , and μ_2 is the second, namely $\mu_2 = (\Delta I_T^d)^2$. If N is not large, there are correction terms of order $1/N$.

The computation of the fourth moment, μ_4 , can be done by the same method as before, but now involves a product of eight fields rather than four. As in the earlier calculation, the expectation value of this product factorizes into a sum of products of two-point functions. This involves somewhat tedious algebra, but fortunately, because we are interested in the difference signal, many of these terms cancel. The details are outlined in the [Appendix](#). The result is

$$\mu_4 = \frac{n_0^2 \omega_0^4}{T^2 \tau^2} \left[3 + 3 \frac{\tau}{T} F\left(\frac{T}{\tau}\right) + \frac{\tau}{T n_0} \right]. \quad (31)$$

Since $(\Delta I_T^d)^2$ is proportional to n_0 , the accuracy with which we can estimate n_0 by this procedure is

$$\frac{\Delta n_0}{n_0} = \frac{\Delta[(\Delta I_T^d)^2]}{(\Delta I_T^d)^2} = \frac{1}{\sqrt{N}} \left[2 + 3 \frac{\tau}{T} F\left(\frac{T}{\tau}\right) + \frac{\tau}{T n_0} \right]^{1/2}. \quad (32)$$

An important advantage of the difference signal is that data in non-overlapping time periods are uncorrelated, because the correlation function $\langle I_T^d(t) I_T^d(0) \rangle$ is proportional to a delta function as can easily be checked by examining the integrand

of the linear term in Equation (14) which is the only one to survive the “differencing”. So this formula should apply even if the basic time interval T is very short. The situation in which the improvement over the direct measurement is most significant is when $1/\omega_0 < T < \tau$. For small T/τ the value of F is approximately T/τ , so we obtain

$$\frac{\Delta n_0}{n_0} \approx \sqrt{\frac{1}{N} \left(5 + \frac{\tau}{n_0 T} \right)}. \quad (33)$$

This accuracy must be compared to the value (19) for direct observations of the incident flux without the aid of the beam splitter.

Indeed, if a direct observation is made over the long time NT , then the accuracy is given by

$$\frac{\Delta n_0}{n_0} = \frac{\Delta I_{NT}}{\bar{I}} = \left(\frac{\sqrt{\pi} \tau}{NT} \right)^{1/2}. \quad (34)$$

The ratio of the uncertainties given by the two methods, (32) and (34), is

$$\eta = \frac{(\Delta n_0)_{\text{split}}^2}{(\Delta n_0)_{\text{dir}}^2} = \frac{1}{\sqrt{\pi}} \left[2x + 3F(x) + \frac{1}{n_0} \right], \quad x = \frac{T}{\tau}. \quad (35)$$

It is a strictly increasing function of x , so its minimum value occurs for x as small as possible. Ideally, $x = T/\tau$ should be of order $1/n_0$ or smaller. This corresponds to an interval T in which of order one photon is detected, $\bar{n}_T \approx 1$. Then the ratio of accuracies can be of order $1/\sqrt{n_0}$, generally a very small number for radio observations, i.e., the improvement in the signal-to-noise ratio over and above the conventional method that does not use any beam splitter is by the factor $\sqrt{n_0} \gg 1$. It can readily be verified that for all intervals $T > 1/\omega_0$ the Heisenberg Uncertainty Principle $T\Delta E \gtrsim 1$ is satisfied. However, because the basic approximation underlying all calculations in this paper is $T \gg 1/\omega_0$ (see Equations (9) and (11)) it is necessary to choose a sufficiently small spectral filter bandwidth to ensure that the $\bar{n}_T \approx 1$ occurs *before* T falls below $1/\omega_0$. To elaborate, since n_0 is independent of the bandwidth and a narrower bandpass means larger τ , one gets a smaller mean number of arriving photons in a given T .

As a concrete example, we take the Arecibo telescope where the system noise temperature is ~ 35 K and the gain is $G \approx 10$ K Jy $^{-1}$ at $\nu = 1$ GHz (1 Jansky = 10^{-26} W m $^{-2}$ Hz $^{-1}$), which converts to the flux density of 3.7 Jy, or 3.7×10^3 photons s $^{-1}$ Hz $^{-1}$ over the entire telescope’s collecting aperture of 300 m diameter. Assuming 50% detection efficiency, the occupation number⁴ of background radio photons set by Equation (8) at

$$n_0 = 210 \quad \text{at } G = 10 \text{ KJy}^{-1}. \quad (36)$$

Thus, by coupling the telescope output with the system presented here the 1σ error in one’s knowledge of the background

⁴ The quantity $1/n_0$ in (35) should strictly speaking be $1/n_0 + 1/n'_0$, where n_0 and $n'_0 \gg n_0$ are, respectively, the number of photons in the radiation of the source and background (including the thermal noise of the amplifier) before and after amplification. To elaborate, the $1/n'_0$ term should be there even if it is totally negligible, as this paper is primarily about the amplified radiation. But the floor of η in Equation (35) should be set by $1/n_0$, which gives the intrinsic shot noise before the signal is amplified. Looking from another viewpoint, due to intrinsic fluctuations the amplified radiation cannot be taken as steady on timescales corresponding to such small values of x that all the x -dependent terms of η have become insignificant.

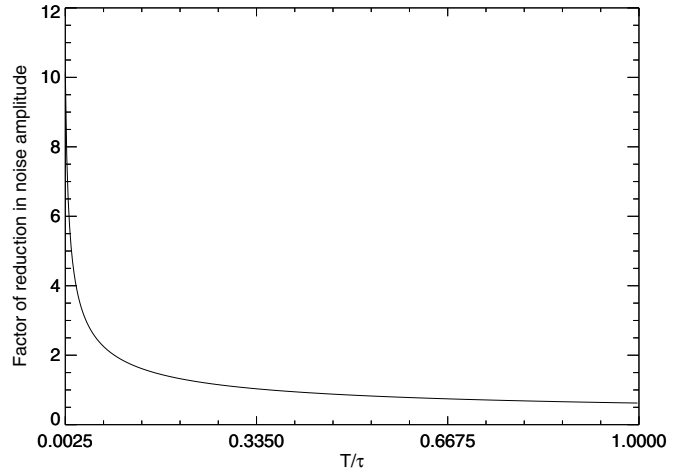


Figure 2. Reduction factor $\eta^{-1/2}$ in noise amplitude attainable by the split beam subtraction as compared to the conventional technique is plotted against the time resolution x , where η and x are defined in Equation (35). The signal is taken from the output of the Arecibo Observatory at central frequency $\nu_0 = 1$ GHz and with $n_0 \approx 210$. The bandpass filter employed is $\delta\nu/\nu_0 \approx 1/300$, and the minimum sampling resolution is $T \approx 0.16$ ns or $x = T/\tau \approx 1/300$.

level may be reduced by as much as the factor $\sqrt{n_0} \approx 14.5$ for equivalent exposures T to a source, i.e., sources fainter than the detection threshold by as much as 14.5 times (for conventional techniques without employing the present scheme) at a given exposure may now in principle be discernible above the background.

Note that because in radio astronomy the data are background dominated, n_0 remains $\gg 1$ irrespective of the source brightness, and thus from the analysis of the previous paragraph the improvement in signal-to-noise is always significant. Moreover, because fast timing is the key to obtaining maximum effect, the proposed technique has application to the search for radio bursts (Thornton et al. 2013) of low brightness. From Equations (36) and (7) the necessary criteria for reaching the maximum signal-to-noise improvement factor of 14.5, viz., $\bar{n}_T \approx 1$ and $T \gg 1/\omega_0$ (see the previous paragraph), are fulfilled by choosing a spectral filter bandpass $\delta\omega < \omega_0/210$.

A point of clarification is in order here. The reader should not be led to believe that the factor $\sqrt{n_0}$ means the choice of a higher equivalent noise temperature (hence n_0) via the use of less effective cryogenics will result in a more sensitive observation. The reason is that in order to detect a weak source of given brightness by the conventional method (34) the exposure time NT has to increase with the background n_0 as $NT \propto n_0^2$, whereas the present method as applied to the $T \ll \tau$ regime would require the less severe $NT \propto n_0$ from Equation (33), i.e., under either scenario the exposure time is always less for smaller n_0 . The other disadvantage of a high n_0 is that one needs faster electronics to detect down at the $\bar{n}_T \approx 1$ limit, and narrower bandpass to keep T above $1/\omega_0$; both are more technically challenging.

5. A SPECIFIC OBSERVATIONAL SETUP

In terms of a realistic design, the receiver sensitivity should not be a problem to detect as few as ~ 10 photons at 1 GHz on the timescale of 1 ns, and the pre-amplifiers and square-law detectors should be fast enough to capture the waveforms within 1 ns. Practically, the feasibility of the proposed scheme hinges on whether one can create a “beam splitter” of radio waves that offers a stable power splitting ratio. The 180° hybrid junctions

appear to be one of the microwave engineering techniques capable of addressing this need (Pożar 2012). Such junctions can be fabricated into waveguide, coax, or planar forms and can be inserted into the signal paths of existing radio telescopes, after the microwave preamplifiers but before the square-law detectors. Assuming the radio wave signals at this point are already converted into guided modes, such an addition should only involve minor system modifications.

For a possible working scenario, let us consider $\nu_0 = 1$ GHz radiation from the output of the Arecibo Telescope, the details of which were presented at the end of the last section. If a bandpass filter of $\delta\nu \approx 3$ MHz is applied, and the highest timing resolution in intensity sampling is pushed right to the limit $T = 1/\omega_0 \approx 0.16$ ns, the parameter x in Equation (35) with τ defined in Equation (7) will have the minimum value of $x \approx 1/300$. For sampling at this rate or lower, the factor of reduction in the noise *amplitude*, as given by $\eta^{-1/2}$ with η defined in Equation (35), is plotted against x in Figure 2. It can be seen that at the limiting resolution the improvement over conventional sensitivity is by an order of magnitude.

6. CONCLUSION

The data gathered by radio telescopes in the frequency range 1–10 GHz are for the majority of sources background dominated by phase noise. By means of a beam splitter it is possible to remove this noise, leaving behind the much weaker fluctuations of the shot (Poisson) noise from which the incident flux can be inferred with an accuracy exceeding the conventional method of analyzing the direct beam. The maximum improvement in sensitivity is of order $\sqrt{n_0}$ where n_0 is the photon occupation number, and is attained when the time resolution is finer than the coherence time, reaching the limit of one arriving photon per resolution element on average. Since radio frequencies are low, the fast timing devices necessary to put this idea into action are relatively easy to build.

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APPENDIX

COMPUTATION OF FOURTH MOMENT

To compute the variance of the variance of I_T^d we require its fourth central moment μ_4 . Now

$$\begin{aligned} \hat{I}_T^d &= \frac{2A}{T} \int_0^T dt_1 [\hat{E}^{t(-)}(t_1)\hat{E}^{t(+)}(t_1) - \hat{E}^{r(-)}(t_1)\hat{E}^{r(+)}(t_1)] \\ &= \frac{1}{2\pi T} \int_0^T dt_1 \int d\omega_1 d\omega'_1 \sqrt{\omega_1 \omega'_1} e^{i(\omega_1 - \omega'_1)t} \\ &\quad \times [\hat{a}^{\dagger\dagger}(\omega_1)\hat{a}^{\dagger}(\omega'_1) - \hat{a}^{\dagger\dagger}(\omega_1)\hat{a}^{\dagger}(\omega'_1)]. \end{aligned} \quad (\text{A1})$$

We need to compute the expectation value of the product of four these operators, for which we will replace the subscript 1 by 2, 3, 4. This involves the expectation value of a product of eight \hat{a} and \hat{a}^\dagger operators, which we may denote in a self-explanatory notation by the symbol $\langle 11'22'33'44' \rangle$. As before, for a Gaussian field, this expectation value factorizes into a sum of products of two-point functions, e.g., $\langle 11' \rangle \langle 23' \rangle \langle 34' \rangle \langle 2'4' \rangle$. There are 24 terms in this sum, corresponding to the different ways of pairing the four \hat{a}^\dagger operators with the four \hat{a} operators.

Each can be associated with the corresponding permutation of the primed symbols, e.g., in this case, using the cycle notation the permutation (1)(234). Each of the 24 terms itself consists of a sum of 16 terms, representing the t or r choices for each pair of fields. Fortunately, as we shall see, many of these terms cancel out.

First, any term containing a pairing such as $\langle 11' \rangle$ will vanish because the t and r terms cancel, as in the expectation value of \hat{I}_T^d . So we can drop all permutations containing (1) for example. Thus we only need to consider permutations belonging to the partitions $[2^2]$ or $[4]$. Any of the three $[2^2]$ permutations, for example (13)(24), can easily be seen to give a result that is simply $\langle (\hat{I}_T^d)^2 \rangle$. So we are left with permutations of class $[4]$.

Next, for any pairing that is in normal order, such as $\langle 12' \rangle$, this factor has the same magnitude irrespective of t or r labels, namely $(1/2)n(\omega_1)\delta(\omega_1 - \omega'_2)$, but terms that differ only by replacing one label t by r or vice versa appear with opposite signs. Thus, if for any of the digits, both the relevant pairs are in normal order, for example for the digit 2, $\langle 12' \rangle$ and $\langle 24' \rangle$, then it is clear that any two terms in the 16 that differ by replacing t by r for the 2, 2' factors will be identical except for sign, and therefore cancel. It is not difficult to see that among the permutations of class $[4]$, these conditions eliminate all but three, namely (1324), (1423), and (1432).

For pairings that are not in normal order, such as $\langle 1'2 \rangle$, the situation is more complicated. We have

$$\langle 1'2 \rangle^{tt} = \langle 1'2 \rangle^{rr} = \left[\frac{1}{2}n(\omega_1) + 1 \right] \delta(\omega'_1 - \omega_2), \quad (\text{A2})$$

but

$$\langle 1'2 \rangle^{tr} = \langle 1'2 \rangle^{rt} = \frac{1}{2}n(\omega_1)\delta(\omega'_1 - \omega_2). \quad (\text{A3})$$

Thus the cancellations are incomplete. For (1324), where two of the factors are in normal order and two in antinormal order we obtain for the sum of all 16 choices of t and r the expression

$$\begin{aligned} \langle 13' \rangle \langle 2'3 \rangle \langle 24' \rangle \langle 1'4 \rangle &= n(\omega_1)n(\omega_2)\delta(\omega_1 - \omega'_3)\delta(\omega'_2 - \omega_3)\delta \\ &\quad \times (\omega_2 - \omega'_4)\delta(\omega'_1 - \omega_4). \end{aligned} \quad (\text{A4})$$

In the contribution of this term to the integral representing μ_4 we can use the delta functions to perform the ω'_j integrations and obtain

$$\begin{aligned} \frac{1}{(2\pi)^4 T^4} \int_0^T dt_1 dt_2 dt_3 dt_4 \int d\omega_1 d\omega_2 d\omega_3 d\omega_4 \\ \times \omega_1 \omega_2 \omega_3 \omega_4 n(\omega_1) n(\omega_2) e^{i[(\omega_1 - \omega_4)t_1 + (\omega_2 - \omega_3)t_2 + (\omega_3 - \omega_1)t_3 + i(\omega_4 - \omega_2)t_4]}. \end{aligned} \quad (\text{A5})$$

As before, the ω_j factors can all be replaced to a very good approximation by ω_0 . In the ω_3 and ω_4 integrations, integrating by parts, these yield extra factors of ω_j or equivalently ω_0 , together with delta functions $\delta(t_3 - t_2)\delta(t_4 - t_1)$. Thus we obtain

$$\begin{aligned} \frac{\omega_0^4}{(2\pi)^2 T^4} \int_0^T dt_1 dt_2 \int d\omega_1 d\omega_2 n(\omega_1) n(\omega_2) e^{i(\omega_1 - \omega_2)(t_1 - t_2)} \\ = \frac{1}{(2\pi)^2 T^4} \int_{-T}^T dt (T - |t|) \left| \int d\omega \omega^2 n(\omega) e^{i\omega t} \right|^2 \\ = \frac{\omega_0^4 n_0^2}{T^4 \tau^2} 2 \int_0^T dt (T - t) e^{-t^2/\tau^2} = \frac{\omega_0^4 n_0^2}{T^3 \tau} F\left(\frac{T}{\tau}\right), \end{aligned} \quad (\text{A6})$$

where $F(x)$ is the same function defined in (16).

For (1423), the calculation is exactly the same apart from relabeling of variables, and the answer is also the same.

Finally we are left with (1432), where *three* of the pairs are in antinormal order. In this case, it turns out that we have not only the same terms involving two n factors, but also one involving just one. We get for the sum of the sixteen terms with all possible choices of t and r ,

$$\langle 14' \rangle \langle 3'4 \rangle \langle 2'3 \rangle \langle 1'2 \rangle = n(\omega_1)[n(\omega_3) + 1]\delta(\omega_1 - \omega'_4)\delta(\omega'_3 - \omega_4)\delta(\omega'_2 - \omega_3)\delta(\omega'_1 - \omega_2). \quad (\text{A7})$$

The term proportional to $n(\omega_3)$ contributes exactly the same as for the two preceding terms, but we now have an extra term arising from the (+1). This is very similar to the extra shot-noise term that appeared in (15). The net contribution to μ_4 is

$$\frac{\omega_0^4}{T^3\tau} \left[n_0^2 F\left(\frac{T}{\tau}\right) + n_0 \right]. \quad (\text{A8})$$

Finally, if we put all these terms together, we obtain

$$\mu_4 = \frac{n_0^2\omega_0^4}{T^2\tau^2} \left[3 + 3\frac{\tau}{T} F\left(\frac{T}{\tau}\right) + \frac{\tau}{Tn_0} \right]. \quad (\text{A9})$$

REFERENCES

- Burke, B. F., & Graham-Smith, F. 2010, *An Introduction to Radio Astronomy* (3rd ed.; Cambridge: Cambridge Univ. Press)
- Birney, D. S., Gonzalez, G., & Oesper, D. 2006, *Observational Astronomy* (2nd ed.; Cambridge: Cambridge Univ. Press)
- Christiansen, W. N., & Högbom, J. A. 1985, *Radio Telescopes* (2nd ed.; Cambridge: Cambridge Univ. Press),
- Hanbury Brown, R., & Twiss, R. Q. 1957, *RSPSA*, **242**, 300
- Pozar, D. M. 2012, *Microwave Engineering* (4th ed.; New York: Wiley)
- Thornton, D., Stappers, B., Bales, M., et al. 2013, *Sci*, **341**, 53